Solving (Quantified) Horn Clauses

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joint work with

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Program Verification and Synthesis as (Horn) Constraint Solving

- Universal temporal properties, e.g., LTL
- Quantifier free invariants/auxiliary assertions
- Proof rule formalization as Horn clauses

Transition System

- v - program variables
- $init(v)$ - initial states
- $step(v, v')$ - transition relation
- $safe(v)$ - safe states

Safety and Termination (WF) of Transition System

$\exists inv(v):$

$$init(v) \rightarrow inv(v)$$
$$inv(v) \wedge step(v, v') \rightarrow inv(v')$$

Safety and Termination (WF) of Transition System

$\exists inv(v)$:

$$init(v) \rightarrow inv(v)$$
$$inv(v) \wedge step(v, v') \rightarrow inv(v')$$
$$inv(v) \rightarrow safe(v) \qquad \text{safety}$$
$$wf(inv(v) \wedge step(v, v')) \qquad \text{well-foundedness}$$

From WF to DWF

$wf(rel(v, v'))$

iff

$\exists ti(v, v')$:

$rel(v, v') \rightarrow ti(v, v')$

$ti(v, v') \wedge rel(v', v'') \rightarrow ti(v, v'')$

$dwf(ti(v, v'))$ disjunctive well-foundedness

Program with procedures

- v - program variables
- $init(v)$ - initial states of main procedure
- $step(v, v')$ - intra-procedural transition relation
- $safe(v)$ - safe states

Program with procedures

- v - program variables
- $init(v)$ - initial states of main procedure
- $step(v, v')$ - intra-procedural transition relation
- $safe(v)$ - safe states
- $call(v, v')$ - parameter passing relation
- $ret(v, v')$ - return value passing

Safety and Termination of Program with Procedures

$\exists sum(v, v')$:

$$init(v_0) \rightarrow sum(v_0, v_0)$$
$$sum(v_0, v_1) \wedge step(v_1, v_2) \rightarrow sum(v_0, v_2)$$
$$sum(v_0, v_1) \wedge call(v_1, v_2) \rightarrow sum(v_2, v_2)$$
$$sum(v_0, v_1) \wedge call(v_1, v_2) \wedge sum(v_2, v_3) \wedge ret(v_3, v_4) \rightarrow sum(v_0, v_4)$$

Safety and Termination of Program with Procedures

$\exists sum(v, v')$:

$$init(v_0) \rightarrow sum(v_0, v_0)$$
$$sum(v_0, v_1) \wedge step(v_1, v_2) \rightarrow sum(v_0, v_2)$$
$$sum(v_0, v_1) \wedge call(v_1, v_2) \rightarrow sum(v_2, v_2)$$
$$sum(v_0, v_1) \wedge call(v_1, v_2) \wedge sum(v_2, v_3) \wedge ret(v_3, v_4) \rightarrow sum(v_0, v_4)$$
$$sum(v_0, v_1) \rightarrow safe(v_1)$$
$$wf(\exists v_1 : sum(v_0, v_1) \wedge call(v_1, v_2))$$

Multi-Threaded Program

- $v = (g, l_1, l_2)$ - global and thread-local variables
- $init(v)$ - initial states
- $safe(v)$ - safe states

Multi-Threaded Program

- $v = (g, l_1, l_2)$ - global and thread-local variables
- $init(v)$ - initial states
- $safe(v)$ - safe states
- $step_1(v, v')$ - transition relation of 1st thread, preserves l_2
- $step_2(v, v')$ - transition relation of 2nd thread, preserves l_1

Rely/Guarantee Rule for Safety

$$\exists inv_1(v)\ \exists inv_2(v)\ \exists env_1(v, v')\ \exists env_2(v, v') :$$

$$init(v) \rightarrow inv_1(v)$$
$$inv_1(v) \wedge step_1(v, v') \rightarrow inv_1(v') \wedge env_2(v, v')$$
$$inv_1(v) \wedge env_1(v, v') \rightarrow inv_1(v')$$
$$\ldots$$
$$inv_1(v) \wedge inv_2(v) \rightarrow safe(v)$$

Clauses for preservation of $inv_2(v)$ are symmetric

Resolving Rely/Guarantee Rule

$$\exists env_2(v, v') :$$

$$\begin{aligned}
&\ldots \\
&inv_1(v) \land step_1(v, v') \rightarrow env_2(v, v') \\
&\ldots \\
&inv_2(v) \land env_2(v, v') \rightarrow inv_2(v') \\
&\ldots
\end{aligned}$$

Into Owicki/Gries Rule

$$\ldots$$

$$env_2(v, v') := inv_1(v) \land step_1(v, v')$$

$$\ldots$$

$$inv_2(v) \land inv_1(v) \land step_1(v, v') \rightarrow inv_2(v')$$

$$\ldots$$

Owicki/Gries Rule for Safety

$$\exists inv_1(v)\ \exists inv_2(v) :$$

$$init(v) \rightarrow inv_1(v)$$
$$inv_1(v) \wedge step_1(v, v') \rightarrow inv_1(v')$$
$$inv_1(v) \wedge inv_2(v) \wedge step_2(v, v') \rightarrow inv_1(v')$$
$$\ldots$$
$$inv_1(v) \wedge inv_2(v) \rightarrow safe(v)$$

Clauses for preservation of $inv_2(v)$ are symmetric

Thread-Modular Rule for Safety

$$\exists \mathit{inv_1}(g, l_1)\ \exists \mathit{inv_2}(g, l_2)\ \exists \mathit{env}(g, g') :$$

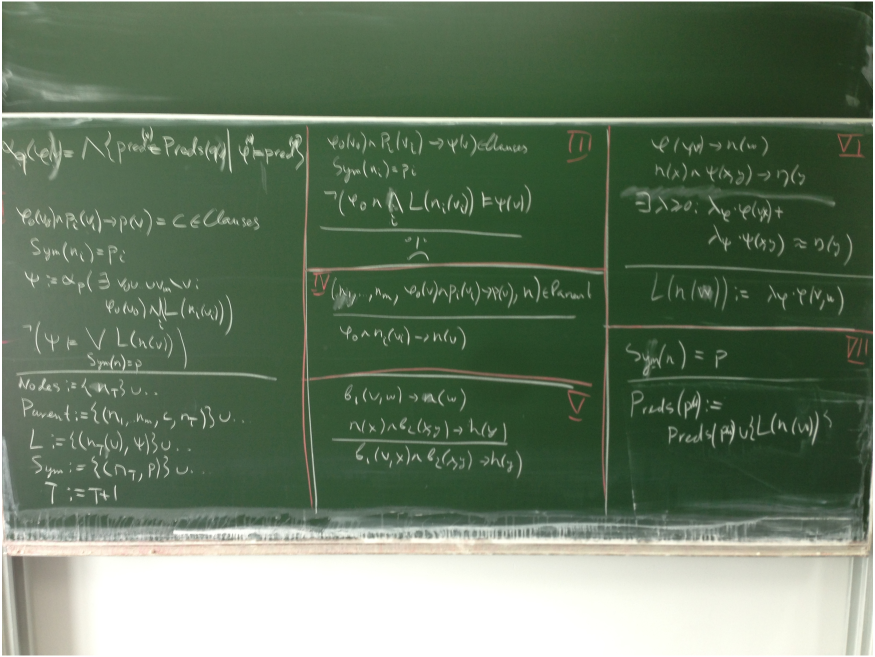
$$\mathit{init}(v) \rightarrow \mathit{inv_1}(g, l_1)$$
$$\mathit{inv_1}(g, l_1) \wedge \mathit{step_1}(v, v') \rightarrow \mathit{inv_1}(g', l_1') \wedge \mathit{env}(g, g')$$
$$\ldots$$
$$\mathit{inv_1}(g, l_1) \wedge \mathit{inv_2}(g, l_2) \rightarrow \mathit{safe}(v)$$

Clauses for preservation of $\mathit{inv_2}(v)$ are symmetric

Quantifier Free Horn Clauses

$$\forall v \; \forall w : \mathit{body}(v, w) \rightarrow \mathit{head}(v)$$

$\mathit{body}(v, w)$ and $\mathit{head}(v)$ are quantifier free



Sym(n_i) = p_i
Nodes := {n_T} ∪ ..
T := T+1
Sym(n) = p

Quantified Horn Clauses

- Existential temporal properties, e.g., CTL
- Program synthesis and infinite-state game solving
- Inference of transactions for concurrent programs

$$\forall v \; \forall w : \mathit{body}(v, w) \rightarrow \exists x : \mathit{head}(v, x)$$

- Quantified invariants/auxiliary assertions

$$\forall v \; \forall w : (\forall y : \mathit{body}(v, w, y)) \rightarrow \mathit{head}(v)$$

Existentially Quantified Horn Clauses

$$\forall v \ \forall w : body(v, w) \rightarrow \exists x : head(v, x)$$

$body(v, w)$ and $head(v, x)$ are quantifier free

Proving CTL Properties

$$(init(v), step(v, v')) \models EF(q(v))$$

$$(init(v), step(v, v')) \models EG(EU(p(v), q(v)))$$

Based on proof system for CTL* by Kesten and Pnueli [TCS'05]

Proving $EF(q(v))$

$$\exists inv(v)\ \exists round(v, v') :$$

$$\begin{aligned}
& init(v) \rightarrow inv(v) \\
& inv(v) \wedge \neg q(v) \rightarrow \exists v' : step(v, v') \\
& \qquad\qquad\qquad\qquad\qquad\quad \wedge\ inv(v') \\
& \qquad\qquad\qquad\qquad\qquad\quad \wedge\ round(v, v') \\
& wf(round(v, v'))
\end{aligned}$$

Decomposing $EG(EU(p(v), q(v)))$

$$(init(v), step(v, v')) \models EG(EU(p(v), q(v)))$$

iff

$\exists mid(v)$:

$$(init(v), step(v, v')) \models EG(mid(v))$$
$$(mid(v), step(v, v')) \models EU(p(v), q(v))$$

Proving $(init(v), step(v, v')) \models EG(mid(v))$ and $(mid(v), step(v, v')) \models EU(p(v), q(v))$

$$\exists mid(v)\ \exists inv_1(v)\ \exists inv_2(v)\ \exists round(v, v') :$$

$$init(v) \rightarrow inv_1(v)$$
$$inv_1(v) \rightarrow mid(v) \wedge \exists v' : step(v, v') \wedge inv_1(v')$$

$$mid(v) \rightarrow inv_2(v)$$
$$inv_2(v) \wedge \neg q(v) \rightarrow p(v) \wedge \exists v' : step(v, v') \wedge inv_2(v') \wedge round(v, v')$$
$$wf(round(v, v'))$$

Solving Infinite-State Game

Given five empty bottles arranged in circle and jar full of water

- Stepmother pours all water from jar into some bottles
- Cinderella empties pair of adjucent bottles
- Jar is refilled for next round

Stepmother wins if some bottle overflows

Formalization of Game Arena

- $v = (v_1, \ldots, v_5)$
- B - bottle volume
- J - jar volume

$$
\begin{aligned}
init(v) \quad &= (v_1 = \cdots = v_5 = 0) \\
cindy(v, v') &= (v'_1 = v'_2 = 0 \wedge same(v_3, v_4, v_5) \vee \\
&\quad \cdots \\
&\quad \vee\ v'_5 = v'_1 = 0 \wedge same(v_2, v_3, v_4)) \\
step(v, v') \ &= (v'_1 \geq v_1 \wedge \cdots \wedge v'_5 \geq v_5 \wedge \\
&\quad v'_1 + \cdots + v'_5 - (v_1 + \cdots + v_5) = J) \\
over(v) \quad &= (v_1 > B \vee \cdots \vee v_5 > B)
\end{aligned}
$$

Stepmother's Victory as Constraint Satisfaction

$$\exists win(v)\ \exists round(v, v') :$$

$$init(v) \rightarrow win(v)$$

$$win(v) \wedge \neg over(v) \wedge cindy(v, v') \rightarrow \exists v'' : step(v', v'') \wedge win(v'') \wedge round(v, v'')$$

$$wf(round(v, v'))$$

Inference of Transactions

- $v = (g, l_1, l_2)$ - global and thread-local variables
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- $v = (g, l_1, l_2)$ - global and thread-local variables
- $init(v)$ - initial states
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- $a(v, v'), b(v, v')$ - transitions of 1st thread
- $p(v, v'), q(v, v')$ - transitions of 2nd thread

Transaction Inference Constraints

- Abstract transitions
- Commutativity
- Transactions
- Invariance

Abstract Transitions: Thread 1

$$inv(v) \land a(v, v') \rightarrow \hat{a}(v, v')$$

Abstract Transitions: Thread 1

$$inv(v) \wedge a(v, v') \rightarrow \hat{a}(v, v')$$

$$inv(v) \wedge \hat{a}(v, v') \wedge b(v', v'') \rightarrow \hat{b}(v', v'')$$

Commutativity: Thread 1

$\hat{a}$ is a right mover and $\hat{b}$ is a left mover

$$inv(v) \wedge \hat{a}(v, v') \wedge \hat{p}(v', v'') \rightarrow \exists v''' : \hat{p}(v, v''') \wedge \hat{a}(v''', v'')$$

Commutativity: Thread 1

$\hat{a}$ is a right mover and $\hat{b}$ is a left mover

$$inv(v) \wedge \hat{a}(v, v') \wedge \hat{p}(v', v'') \rightarrow \exists v''' : \hat{p}(v, v''') \wedge \hat{a}(v''', v'')$$

$$inv(v) \wedge \hat{a}(v, v') \wedge \hat{q}(v', v'') \rightarrow \exists v''' : \hat{q}(v, v''') \wedge \hat{a}(v''', v'')$$

Commutativity: Thread 1

$\hat{a}$ is a right mover and $\hat{b}$ is a left mover

$$inv(v) \wedge \hat{a}(v, v') \wedge \hat{p}(v', v'') \rightarrow \exists v''' : \hat{p}(v, v''') \wedge \hat{a}(v''', v'')$$

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$$inv(v) \wedge \hat{p}(v, v') \wedge \hat{b}(v', v'') \rightarrow \exists v''' : \hat{b}(v, v''') \wedge \hat{p}(v''', v'')$$

Commutativity: Thread 1

$\hat{a}$ is a right mover and $\hat{b}$ is a left mover

$$inv(v) \land \hat{a}(v, v') \land \hat{p}(v', v'') \rightarrow \exists v''' : \hat{p}(v, v''') \land \hat{a}(v''', v'')$$

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$$inv(v) \land \hat{p}(v, v') \land \hat{b}(v', v'') \rightarrow \exists v''' : \hat{b}(v, v''') \land \hat{p}(v''', v'')$$

$$inv(v) \land \hat{q}(v, v') \land \hat{b}(v', v'') \rightarrow \exists v''' : \hat{b}(v, v''') \land \hat{q}(v''', v'')$$

Transactions

$$inv(v) \wedge \hat{a}(v, v') \wedge \hat{b}(v', v'') \to \widehat{ab}(v, v'')$$

Transactions

$$inv(v) \land \hat{a}(v, v') \land \hat{b}(v', v'') \rightarrow \widehat{ab}(v, v'')$$

$$inv(v) \land \hat{p}(v, v') \land \hat{q}(v', v'') \rightarrow \widehat{pq}(v, v'')$$

Invariance

$$init(v) \rightarrow inv(v)$$

Invariance

$$init(v) \rightarrow inv(v)$$

$$inv(v) \wedge \widehat{ab}(v, v') \rightarrow inv(v')$$

Invariance

$$init(v) \rightarrow inv(v)$$

$$inv(v) \land \widehat{ab}(v, v') \rightarrow inv(v')$$

$$inv(v) \land \widehat{pq}(v, v') \rightarrow inv(v')$$

Invariance

$$init(v) \rightarrow inv(v)$$

$$inv(v) \wedge \widehat{ab}(v, v') \rightarrow inv(v')$$

$$inv(v) \wedge \widehat{pq}(v, v') \rightarrow inv(v')$$

$$inv(v) \rightarrow safe(v)$$

Universally Quantified Horn Clauses

$$\forall v \; \forall w : (\forall y : body(v, w, y)) \rightarrow head(v)$$

$body(v, w, y)$ and $head(v)$ are quantifier free

Vefication with Universally Quantified Invariants

```
for(i = 0; i < n; i++) { a[i] = i; }
assert("forall p: i <= p && p < n -> a[p] == p");
```

State-of-the-art recipe (e.g., Gopan et al. POPL'05,Gulwani et al. POPL'08, Halbwachs et al. PLDI'08, Dillig et al. ESOP'10, Logozzo et al. POPL'11, Alberti et al. CAV'12, Larraz et al. VMCAI'13):

- Quantification template
- Instantiation template
- Shape template with abstract domains

Templates for Universally Quantified Invariants

- Quantification template

$$\forall p : inv(i, n, p, a(p))$$

- Instantiation template

$$inv(i, n, e_1, a(e_1)) \wedge \cdots \wedge inv(i, n, e_k, a(e_k))$$

- Shape template with abstract domains

$$inv(i, n, p, a(p)) = (guard(i, n, p) \rightarrow property(i, n, p, a(p)))$$

*under-approximation for $guard(i, n, p)$

Universally Quantified Clauses

```
for(i = 0; i < n; i++) { a[i] = i; }
assert("forall p: i <= p && p < n -> a[p] == p");
```

- Quantification template $\forall p : inv(i, n, p, a(p))$

$$\exists inv(i, n, p, \overbrace{v}^{a(p)}) :$$

$$i = 0 \rightarrow (\forall p : inv(i, n, p, a(p)))$$

$$(\forall p : inv(i, n, p, a(p))) \wedge i < n \wedge i' = i + 1 \wedge a' = a\{i := i\} \rightarrow$$
$$(\forall q : inv(i', n, q, a'(q)))$$

$$(\forall p : inv(i, n, p, a(p))) \rightarrow (\forall q : 0 \leq q < n \rightarrow a(q) = q)$$

Universally Quantified Clauses

```
for(i = 0; i < n; i++) { a[i] = i; }
assert("forall p: i <= p && p < n -> a[p] == p");
```

- Quantification template $\forall p : inv(i, n, p, a(p))$

$$\exists inv(i, n, p, \overbrace{v}^{a(p)}) :$$

$$i = 0 \rightarrow inv(i, n, p, a(p))$$

$$(\forall p : inv(i, n, p, a(p))) \wedge i < n \wedge a' = a\{i := i\} \rightarrow inv(i + 1, n, q, a'(q))$$

$$(\forall p : inv(i, n, p, a(p))) \wedge 0 \leq q < n \rightarrow a(q) = q$$

Quantifier Instantiation Heuristic

Instantiation Constraint Generation

- If $a(p)$ occcurs in clause then $p \in inst(a)$
- If $a' = a\{\cdot := \cdot\}$ occcurs in clause then $inst(a') \subseteq inst(a)$

Example

- Clause

$$(\forall p : inv(i, n, p, a(p))) \land i < n \land a' = a\{i := i\}$$
$$\rightarrow inv(i + 1, n, q, a'(q))$$

- Instantiation constaints $q \in inst(a')$ and $inst(a') \subseteq inst(a)$
- Instantiation solution $inst(a) = inst(a') = \{q\}$

Quantifier Instantiation Validation

$inv(i, n, p, v) = (0 \leq p < i \rightarrow v = p)$ and $inst(a) = inst(a') = \{q\}$

$$
\begin{aligned}
&(0 \leq q < i \rightarrow q = a(q)) \\
&\wedge\ i < n \\
&\wedge\ a'(i) = i \wedge (q \neq i \rightarrow a'(q) = a(q)) \\
&\rightarrow (0 \leq q < i + 1 \rightarrow a'(q) = q)
\end{aligned}
$$

Instantiation Based Proof

$$(0 \leq q < i \rightarrow a(q) = q) \wedge i < n \wedge a'(i) = i \wedge (q \neq i \rightarrow a'(q) = a(q))$$

$$(0 \leq q < i + 1 \rightarrow a'(q) = q)$$

Instantiation Based Proof

$$(0 \leq q < i \rightarrow a(q) = q) \wedge i < n \wedge a'(i) = i \wedge (q \neq i \rightarrow a'(q) = a(q))$$

$$0 \leq q < i + 1$$

$$a'(q) = q$$

$$(0 \leq q < i + 1 \rightarrow a'(q) = q)$$

Instantiation Based Proof

$$(0 \leq q < i \rightarrow a(q) = q) \wedge i < n \wedge a'(i) = i \wedge (q \neq i \rightarrow a'(q) = a(q))$$

$$0 \leq q < i + 1$$

$$q \neq i \qquad\qquad q = i$$

$$a'(q) = q$$

$$(0 \leq q < i + 1 \rightarrow a'(q) = q)$$

Instantiation Based Proof

$$(0 \leq q < i \rightarrow a(q) = q) \wedge i < n \wedge a'(i) = i \wedge (q \neq i \rightarrow a'(q) = a(q))$$

$0 \leq q < i + 1$

$q \neq i$ $\qquad\qquad$ $q = i$

$a(q) = q$

$a'(q) = q$

$(0 \leq q < i + 1 \rightarrow a'(q) = q)$

Instantiation Based Proof

$$(0 \leq q < i \rightarrow a(q) = q) \wedge i < n \wedge a'(i) = i \wedge (q \neq i \rightarrow a'(q) = a(q))$$

$0 \leq q < i + 1$

$q \neq i$ | $q = i$

$a(q) = q$

$a'(q) = a(q)$

$a'(q) = q$

$$(0 \leq q < i + 1 \rightarrow a'(q) = q)$$

Instantiation Based Proof

$$(0 \leq q < i \to a(q) = q) \wedge i < n \wedge a'(i) = i \wedge (q \neq i \to a'(q) = a(q))$$

$$0 \leq q < i + 1$$

$$q \neq i \qquad\qquad\qquad q = i$$

$$a(q) = q$$

$$a'(q) = a(q)$$

$$a'(q) = q$$

$$a'(q) = q$$

$$(0 \leq q < i + 1 \to a'(q) = q)$$

Instantiation Based Proof

$$(0 \leq q < i \rightarrow a(q) = q) \wedge i < n \wedge a'(i) = i \wedge (q \neq i \rightarrow a'(q) = a(q))$$

$0 \leq q < i + 1$

$q \neq i$	$q = i$
$a(q) = q$	$a'(q) = q$
$a'(q) = a(q)$	
$a'(q) = q$	

$a'(q) = q$

$$(0 \leq q < i + 1 \rightarrow a'(q) = q)$$

Universally Quantified Invariant for Termination

```
for(i = 0; i < n; i++) {
  a[i] = 1;
}
while (x > 0) {
  for(i = 0; i < n; i++) {
    x = x-a[i];
  }
}
```

$$
\begin{aligned}
inv_1(i, n, x, p, v) &= (0 \leq p < i \rightarrow v \geq 1) \\
inv_2(i, n, x, p, v) &= (0 \leq x \wedge 0 \leq i < n \wedge (0 \leq p < n \rightarrow v \geq 1)) \\
&= inv_3(i, n, x, p, v)
\end{aligned}
$$

Further Pointers

- Solving recursion-free clauses over LI+UIF, [APLAS'11]
- Solving quantifier free clauses and well-foundedness, [PLDI'12]
- Solving existentially quantified clauses: [CAV'13]
- Solving universally quantified clauses: [SAS'13]
- Proof rules for multi-threaded programs [POPL'11, CAV'11, TACAS'12]
- Proof rules for functional programs [CAV'11, SAS'12]
- Software verification competition [SV-COMP'12, SV-COMP'13]
- Separation logic modulo theories [APLAS'13]
- A Constraint-Based Approach to Solving Games on Infinite Graphs [POPL'2014]