Symbol Elimination for Program Analysis

Laura Kovács

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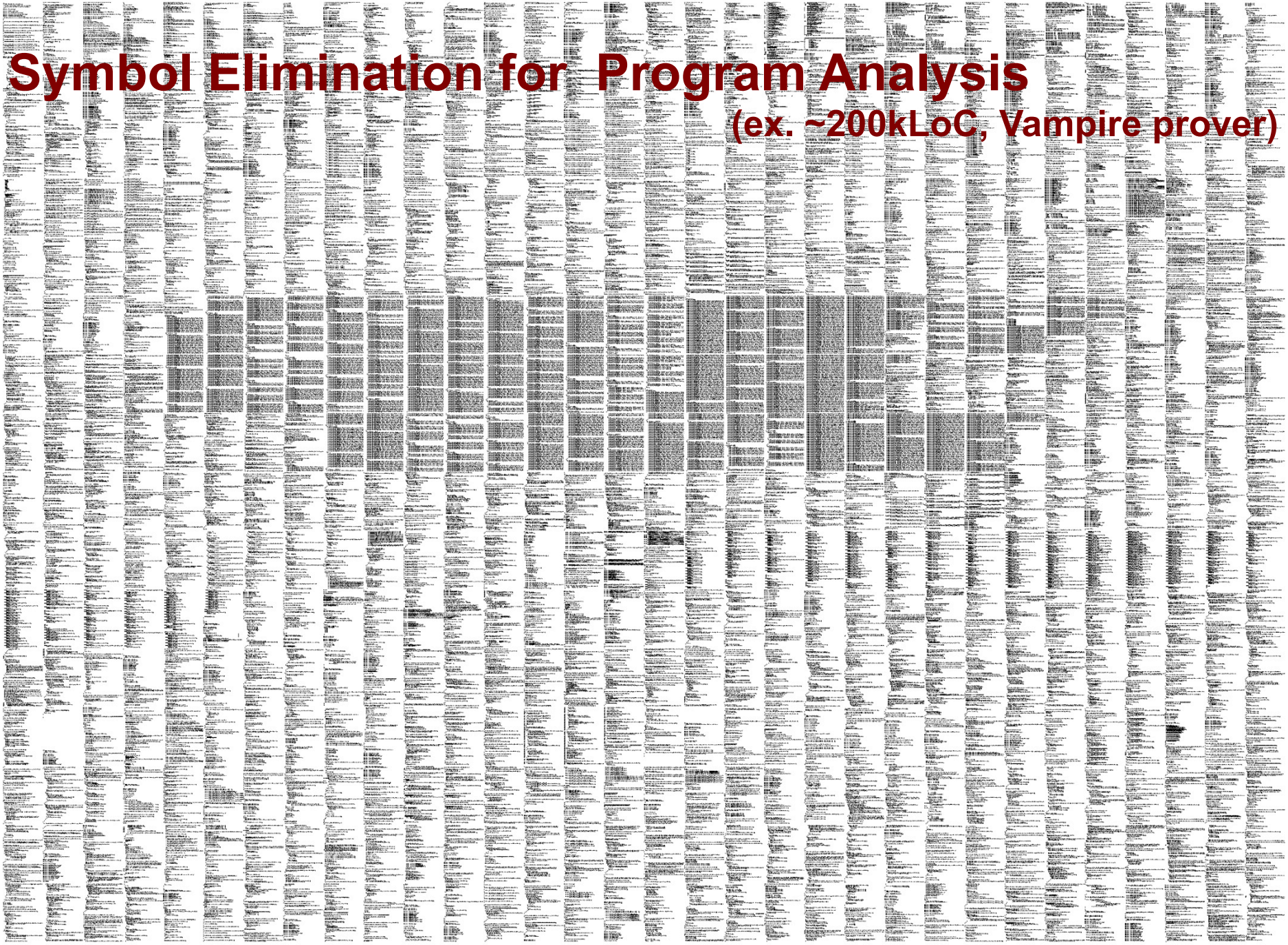
(ex. ~200kLoC, Vampire prover)



Symbol Elimination for Program Analysis

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a=0, b=0, c=0;
while (a<n) do

if A[a]>0 then B[b]=A[a]+h(b); b=b+1;
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Program property:

$$(\forall p)(0 \le p < b \Rightarrow (\exists q)(0 \le q < a \wedge B[p]=A[q]+h(p) \wedge A[q]>0)$$

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while (cnt<n) do

t=fib1; fib1=fib1+fib2; fib2=t; cnt++;

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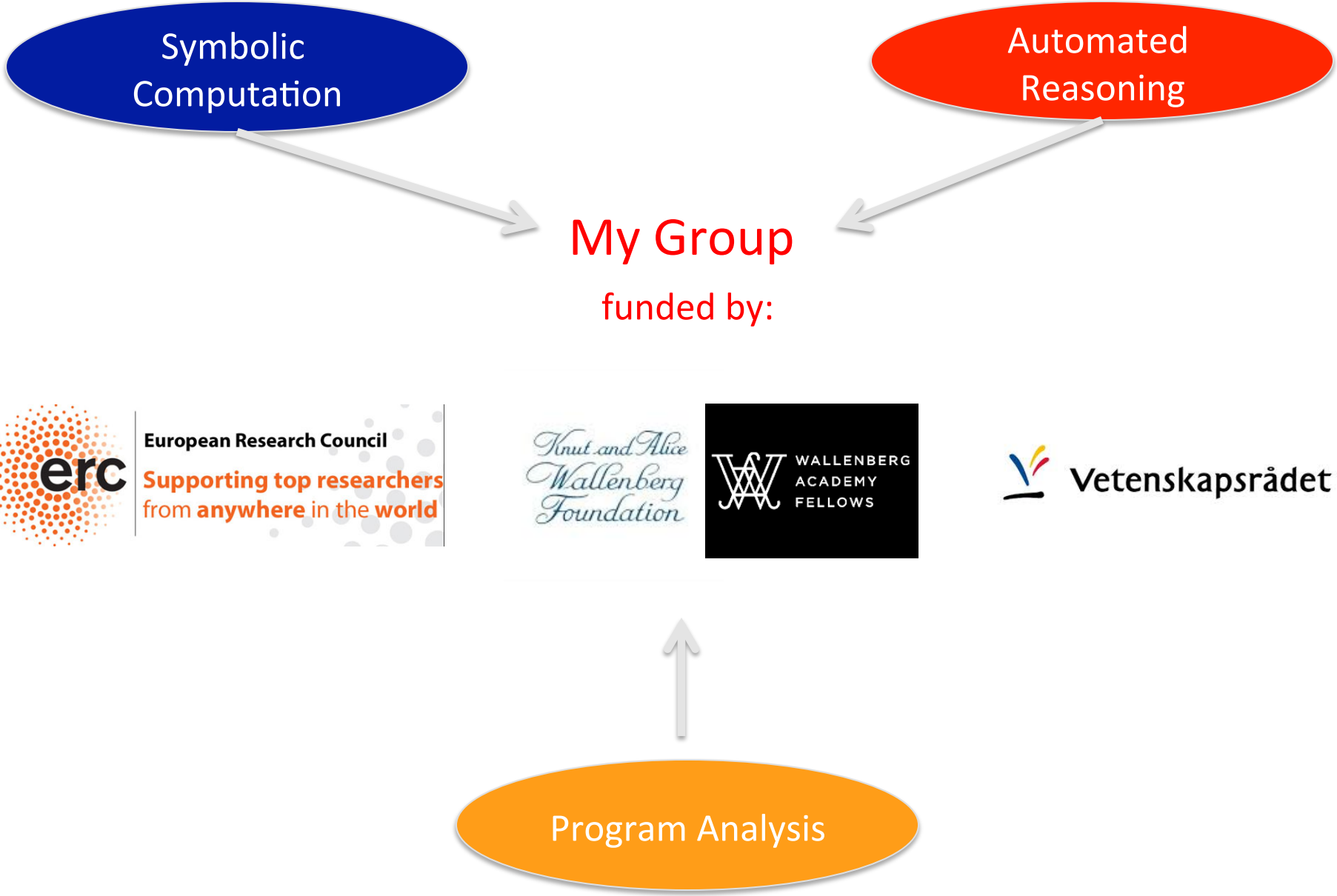
Symbolic Computation
Automated Reasoning
My Group
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Knut and Alice Wallenberg Foundation
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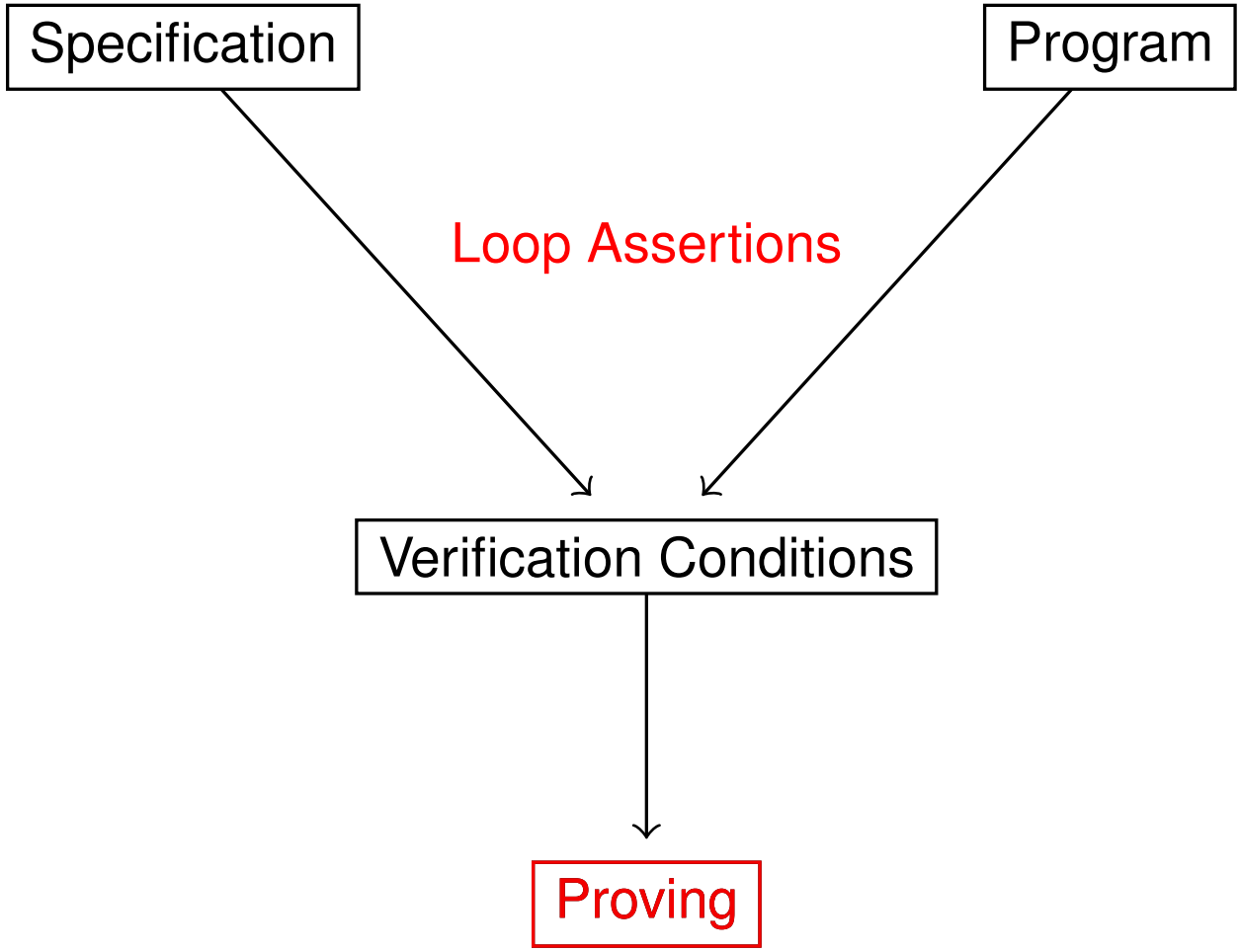
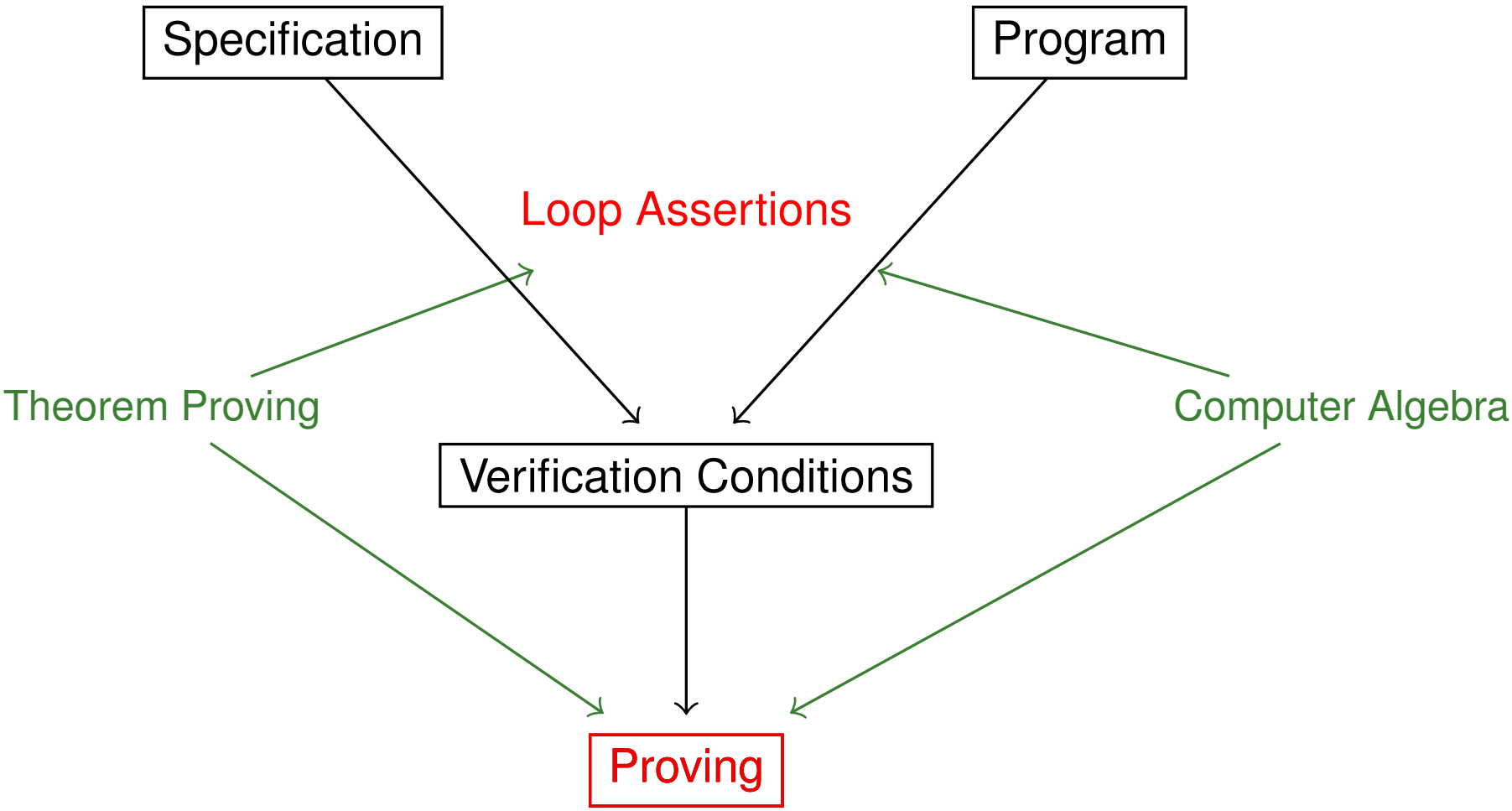
Program Analysis

How do we tackle automated program analysis?

1. Capture relevant aspects of the system formally (using recurrence equations and logic);

2. Design and implement algorithms for analyzing the formal model (computer algebra and theorem proving).


Specification
Program
Loop Assertions
Verification Conditions
Proving


Specification
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Assertion Synthesis — Example: Array Partition

Program

$a := 0;\ b := 0;\ c := 0;$

while $(a < N)$ **do**

  if $A[a] \geq 0$

    then $B[b] := A[a];\ b := b + 1$

    else $C[c] := A[a];\ c := c + 1;$

  $a := a + 1;$

end do

Loop Assertions

$a = b + c$

$a \geq 0 \ \wedge \ b \geq 0 \ \wedge \ c \geq 0$

$a \leq N \quad \vee \quad N \leq 0$

$(\forall p)(p \geq b \Rightarrow B[p] = B_0[p])$

$(\forall p)(0 \leq p < b \Rightarrow$

$\quad B[p] \geq 0 \ \wedge$

$\quad (\exists i)(0 \leq i < a \ \wedge \ A[a] = B[p]))$

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Loop Assertions

Polynomial Equalities and Inequalities, Quantified FO properties

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Our Approach

Loop

Assertions

Our Approach

Our Approach:
SYMBOL ELIMINATION

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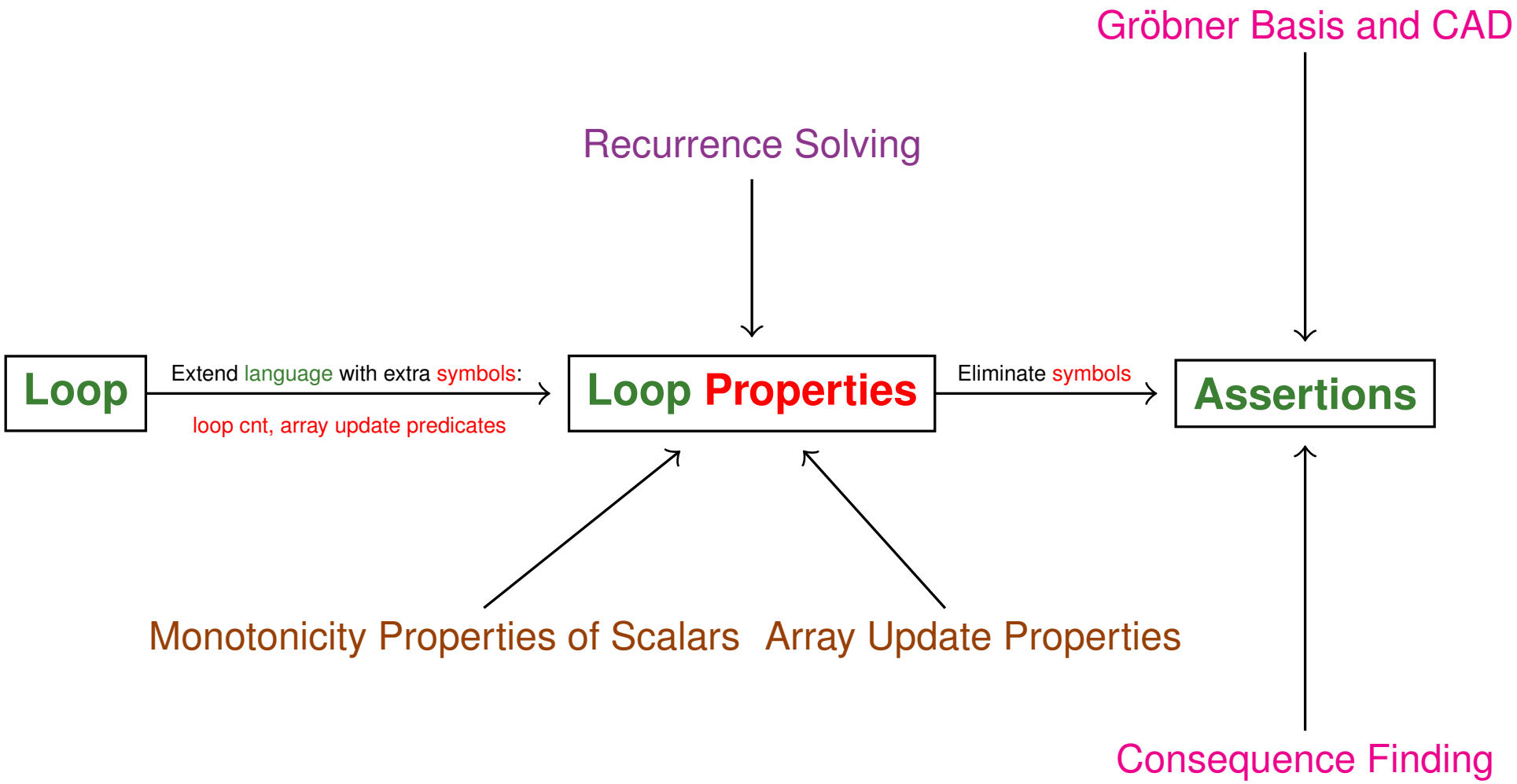

Outline

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Polynomial Invariants

$x := 1;\ y := 0;$

while $\ldots$ **do** $x := 2 * x;\ y := \frac{1}{2} * y + 1$ **end do**

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$$n \geq 0,\ a = 2^n,\ b = 2^{-n}$$

$$\begin{cases} x^{(n+1)} &= 2 * x^{(n)} \\ y^{(n+1)} &= \frac{1}{2} * y^{(n)} + 1 \end{cases}$$

2. Solve recurrence relations $\rightarrow$ closed forms of variables: functions of iteration counter n
   $\uparrow$ methods from symbolic summation;

$$\begin{cases} x^{(n)} &= 2^n * x^{(0)} \\ y^{(n)} &= \frac{1}{2^n} * y^{(0)} - \frac{2}{2^n} + 2 \end{cases}$$

3. Identify polynomial/algebraic dependencies among exponentials in n;

$$\begin{cases} x^{(n)} &= a * x^{(0)} \\ y^{(n)} &= b * y^{(0)} - 2 * b + 2 \\ 0 &= a * b - 1 = 2^n * \frac{1}{2^n} - 1 \end{cases}$$

4. Eliminate n and variables standing for algebraically related exponentials in $n \rightarrow$
   $\uparrow$ elimination by Gröbner bases;

$$x * y - 2 * x + 2 = 0$$

5. Result: Polynomial Ideal $\rightarrow$ Finite basis

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Polynomial Invariants (TACAS08, ISSAC17)

Symbol elimination by Gröbner bases computation:

- Loops with assignments and nested conditionals.

  Structural constraints on assignments with polynomial rhs.

  $\leftarrow$ Recurrence solving (C-finite, Hypergeometric)

  $\leftarrow$ Restricted multiplication among variables;

  Tests are ignored $\rightarrow$ non-deterministic programs

- Automated Loop Invariant Generation by Algebraic Techniques Over the Rationals for P-solvable Loops:

  $\leftarrow$ Sound and complete - termination bound: #vars · #branches;

- Polynomial invariant ideals represented by Gröbner bases;

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- Implementation: `ALIGATOR` https://ahumenberger.github.io/aligator/

Example from Sharon's Talk

while $\ldots$ **do** $x := x + y;\ y := y + 2$ **end do**

Inductive Polynomial Invariant:

$$4x - y^2 + 2y = 4x_0 - y_0^2 + 2 * y_0$$

Outline

Quantified Invariants

$a := 0; b := 0; c := 0;$
while $(a \leq k)$ **do**
  if $A[a] \geq 0$
    then $B[b] := A[a]; b := b + 1;$
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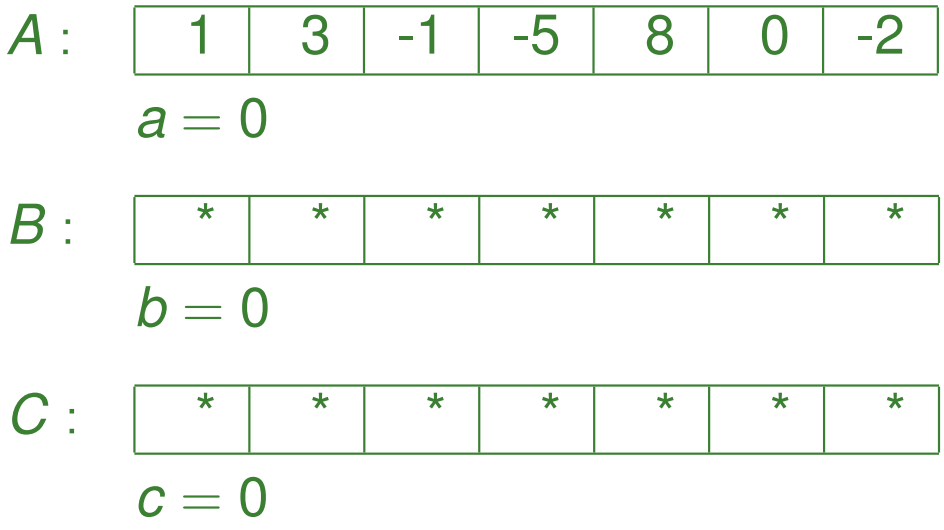
  $a := a + 1;$
end do



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end do

A :

1	3	-1	-5	8	0	-2

$a = 7$

B :

1	3	8	0	*	*	*

$b = 4$

C :

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$c = 3$

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Invariants with $\forall$ $\exists$

- Each of $B[0], \ldots, B[b-1]$ is non-negative and equal to one of $A[0], \ldots, A[a-1]$.

$$(\forall p)(0 \leq p < b \Rightarrow B[p] \geq 0 \wedge (\exists i)(0 \leq i < a \wedge A[i] = B[p]))$$

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Invariants with $\forall$ $\exists$

- Each of $B[0], \ldots, B[b-1]$ is non-negative and equal to one of $A[0], \ldots, A[a-1]$.
- Each of $C[0], \ldots, C[c-1]$ is negative and equal to one of $A[0], \ldots, A[a-1]$.

Invariants with $\forall$

- For every $p \geq b$, the value of $B[p]$ is equal to its initial value.
- For every $p \geq c$, the value of $C[p]$ is equal to its initial value.

Invariants and Symbol Elimination: The General Picture

Quant. Invariants and Symbol Elimination

```
a := 0; b := 0; c := 0;
while (a ≤ k) do
  if A[a] ≥ 0
    then B[b] := A[a];b := b + 1;
    else C[c] := A[a];c := c + 1;
  a := a + 1;
end do
```

1. Extend the language $\mathcal{L}$ to $\mathcal{L}'$:
   - loop counter n;
   - variables as functions of n: $v^{(i)}$ with $0 \leq i < n$
   - loop property predicates: $iter, upd_V(i,p), upd_V(i,p,x)$

$$(\forall i)(i \in iter \Leftrightarrow 0 \leq i \wedge i < n)$$

$$upd_B(i,p) \Leftrightarrow i \in iter \wedge p = b^{(i)} \wedge A[a^{(i)}] \geq 0$$

$$upd_B(i,p,x) \Leftrightarrow upd_B(i,p) \wedge x = A[a^{(i)}]$$

$$a = b + c,\ a \geq 0,\ b \geq 0,\ c \geq 0$$

$$(\forall i \in iter)(a^{(i+1)} > a^{(i)})$$

$$(\forall i \in iter)(b^{(i+1)} = b^{(i)} \vee b^{(i+1)} = b^{(i)} + 1)$$

$$(\forall i \in iter)(a^{(i)} = a^{(0)} + i)$$

$$(\forall j,k \in iter)(k \geq j \Rightarrow b^{(k)} \geq b^{(j)})$$

$$(\forall j,k \in iter)(k \geq j \Rightarrow b^{(j)} + k \geq b^{(k)} + j)$$

$$(\forall p)(b^{(0)} \leq p < b^{(n)} \Rightarrow (\exists i \in iter)(b^{(i)} = p \wedge A[a^{(i)}] \geq 0))$$

$$(\forall i)\neg upd_B(i,p) \Rightarrow B^{(n)}[p] = B^{(0)}[p]$$

$$upd_B(i,p,x) \wedge (\forall j > i)\neg upd_B(j,p) \Rightarrow B^{(n)}[p] = x$$

$$(\forall i \in iter)(A[a^{(i)}] \geq 0 \Rightarrow B^{(i+1)}[b^{(i)}] = A[a^{(i)}] \wedge b^{(i+1)} = b^{(i)} + 1 \wedge c^{(i+1)} = c^{(i)})$$

Quant. Invariants and Symbol Elimination

```
a := 0; b := 0; c := 0;
while (a ≤ k) do
  if A[a] ≥ 0
    then B[b] := A[a];b := b + 1;
    else C[c] := A[a];c := c + 1;
  a := a + 1;
end do
```

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2. Collect loop properties in $\mathcal{L}'$:
   - Polynomial scalar properties
   - Monotonicity properties of scalars
   - Update predicates of arrays
   - Translation of guarded assignments

$$a = b + c,\ a \geq 0,\ b \geq 0,\ c \geq 0$$
$$(\forall i \in iter)(a^{(i+1)} > a^{(i)})$$
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$$(\forall i)\neg upd_B(i,p) \Rightarrow B^{(n)}[p] = B^{(0)}[p]$$
$$upd_B(i,p,x) \wedge (\forall j > i)\neg upd_B(j,p) \Rightarrow B^{(n)}[p] = x$$

3. Eliminate symbols $\rightarrow$ Invariants

$$(\forall i \in iter)(A[a^{(i)}] \geq 0 \Rightarrow B^{(i+1)}[b^{(i)}] = A[a^{(i)}] \wedge b^{(i+1)} = b^{(i)} + 1 \wedge c^{(i+1)} = c^{(i)})$$

Quant. Invariants and Symbol Elimination

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while (a ≤ k) do
  if A[a] ≥ 0
    then B[b] := A[a];b := b + 1;
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$$(\forall j,k \in iter)(k \geq j \Rightarrow b^{(j)} + k \geq b^{(k)} + j)$$
$$(\forall p)(b^{(0)} \leq p < b^{(n)} \Rightarrow (\exists i \in iter)(b^{(i)} = p \wedge A[a^{(i)}] \geq 0))$$
$$(\forall i)\neg upd_B(i,p) \Rightarrow B^{(n)}[p] = B^{(0)}[p]$$
$$upd_B(i,p,x) \wedge (\forall j > i)\neg upd_B(j,p) \Rightarrow B^{(n)}[p] = x$$
$$(\forall i \in iter)(A[a^{(i)}] \geq 0 \Rightarrow B^{(i+1)}[b^{(i)}] = A[a^{(i)}] \wedge b^{(i+1)} = b^{(i)} + 1 \wedge c^{(i+1)} = c^{(i)})$$

Quant. Invariants and Symbol Elimination

```
a := 0; b := 0; c := 0;
while (a ≤ k) do
  if A[a] ≥ 0
    then B[b] := A[a];b := b + 1;
    else C[c] := A[a];c := c + 1;
  a := a + 1;
end do
```

1. Extend the language $\mathcal{L}$ to $\mathcal{L}'$:
   - loop counter n;
   - variables as functions of n: $v^{(i)}$ with $0 \leq i < n$
   - loop property predicates: $iter, upd_V(i,p), upd_V(i,p,x)$

$$(\forall i)(i \in iter \Leftrightarrow 0 \leq i \wedge i < n)$$
$$upd_B(i,p) \Leftrightarrow i \in iter \wedge p = b^{(i)} \wedge A[a^{(i)}] \geq 0$$
$$upd_B(i,p,x) \Leftrightarrow upd_B(i,p) \wedge x = A[a^{(i)}]$$

2. Collect loop properties in $\mathcal{L}'$:
   - Polynomial scalar properties
   - Monotonicity properties of scalars
   - Update predicates of arrays
   - Translation of guarded assignments

$$a = b + c,\ a \geq 0,\ b \geq 0,\ c \geq 0$$
$$(\forall i \in iter)(a^{(i+1)} > a^{(i)})$$
$$(\forall i \in iter)(b^{(i+1)} = b^{(i)} \vee b^{(i+1)} = b^{(i)} + 1)$$
$$(\forall i \in iter)(a^{(i)} = a^{(0)} + i)$$
$$(\forall j,k \in iter)(k \geq j \Rightarrow b^{(k)} \geq b^{(j)})$$
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3. Eliminate symbols $\rightarrow$ Invariants

HOW?

Quantified Invariants and Symbol Elimination

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$$\xrightarrow[\text{Theorem Proving}]{\text{Saturation}} I_1, I_2, I_3, I_4, I_5, \ldots$$

Symbol Elimination by First-Order Theorem Proving

1. Reasoning in first-order theories

$$x \geq y \iff x > y \vee x = y$$
$$x > y \Rightarrow x \neq y$$
$$x \geq y \wedge y \geq z \Rightarrow x \geq z$$
$$x + 1 > x$$
$$x \geq y + 1 \iff x > y$$

2. Procedures for eliminating symbols → INVARIANTS

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- For every loop variable v → TARGET SYMBOLS v_0 and v:

$$v^{(0)} = v_0 \qquad \text{and} \qquad v^{(n)} = v$$

- USABLE symbols:
  - target or interpreted symbols;
  - skolem functions introduced by Vampire;
- Reduction (elimination) ordering $\succ$:
  useless symbols $\succ$ usable symbols.

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Quantified Invariants and Symbol Elimination

(CAV13, LPAR15)

Symbol elimination by saturation-based theorem proving:

(joint work with Y. Chen, S. Robillard and A. Voronkov)

- Express loop properties in a language containing extra symbols;
- Every logical consequence of these properties is a valid loop property, but not an invariant;
- Run a saturation theorem prover for eliminating extra symbols;
- Derived formulas in the language of the loop are loop invariants.

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Implementation: `Vampire` http://vprover.org

Symbol Elimination for Program Analysis

New Extensions in Theorem Proving

- First class boolean sort
  - example: boolean arrays, boolean flags;
- Theory of polymorphic arrays
  - example: programs with integer arrays and boolean arrays;
- If-then-Else and Let-in constructs
  - example: partial correctness of programs with nested conditionals.

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Symbol Elimination for Program Analysis (CPP16)

New Extensions in Theorem Proving: **the FOOL Logic**

(joint work with E. Kotelnikov, M. Suda, A. Voronkov)

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- Theory of polymorphic arrays
  - example: programs with integer arrays and boolean arrays;
- If-then-Else and Let-in constructs
  - example: partial correctness of programs with nested conditionals.

Cookies for Program Analysis (CPP16)

A partial correctness statement:

```
{∀X(p(X) => X ≥ 0)}
{∀X(q(X) > 0)}
{p(a)}
if (r[a]) {
  a := a+1
}
else {
  a := a + q(a).
}
{a > 0}
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The next state function for `a`:

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a1 =
  if r[a]
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In Vampire:

```
thf(1,type,p : $int > $o).
thf(2,type,q : $int > $int).
thf(3,type,r : $array($int, $o)).
thf(4,type,a : $int).
thf(5,type,a1: $int).

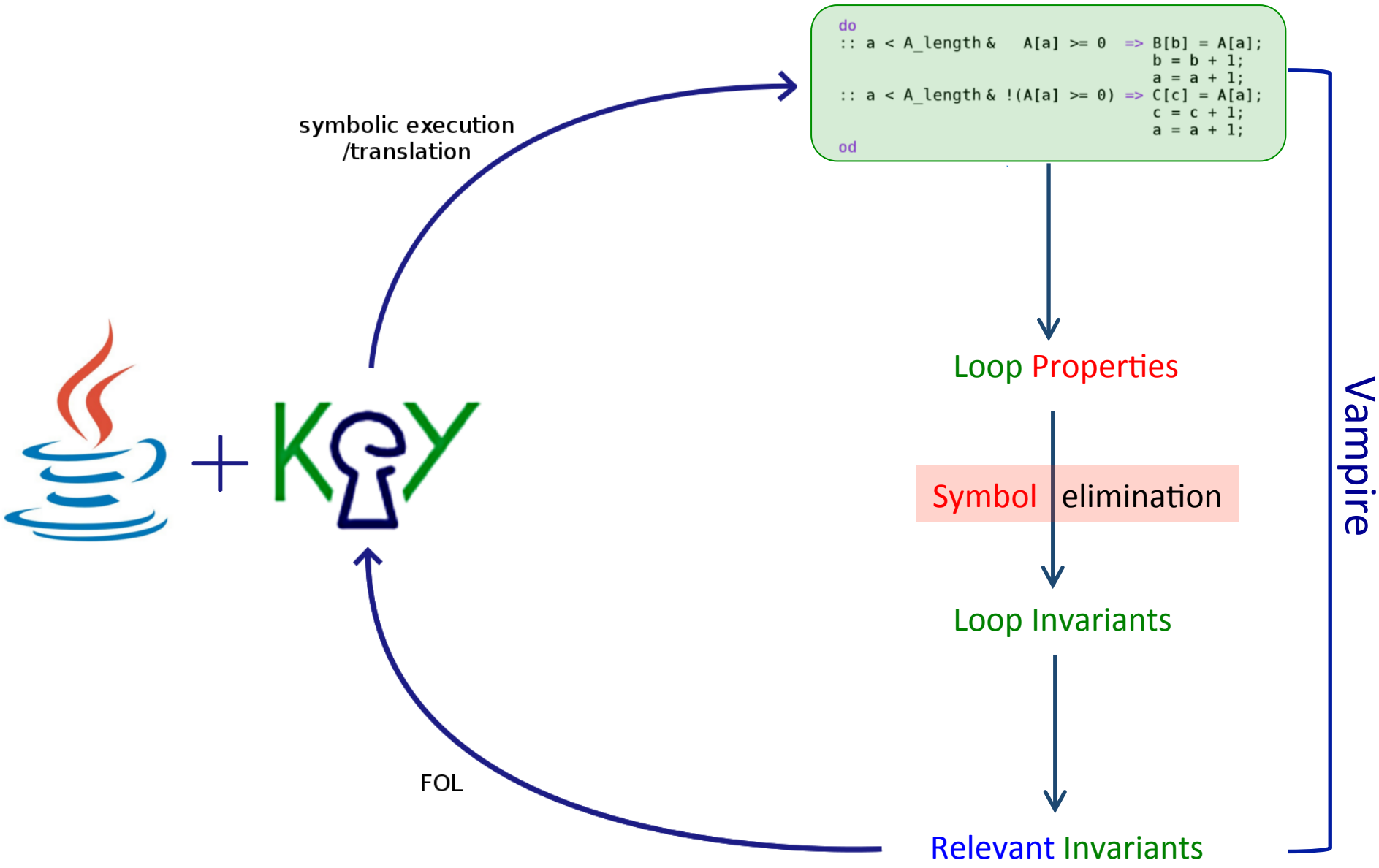
thf(6,hypothesis,! [X:$int] :
  (p(X) => $greatereq(X,0))).
thf(7,hypothesis,! [X:$int] :
  ($greater(q(X),0))).
thf(8,hypothesis,p(a)).

thf(9,hypothesis,
  a1 = $ite($select(r,a),
    $let(a:=$sum(a,1),a),
    $let(a:=$sum(a,q(a)),a)
)).

tff(10,conjecture,$greater(a1,0)).
```

Symbol Elimination in Program Verification (LPAR15)

(joint work with W. Ahrendt and S. Robillard)



Conclusions and Further Work

Symbol Elimination for Program Analysis

- Combines symbolic computation and saturation theorem proving;
- Fully automatic;
- Polynomial properties with quantifier alternations.

Further Work

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