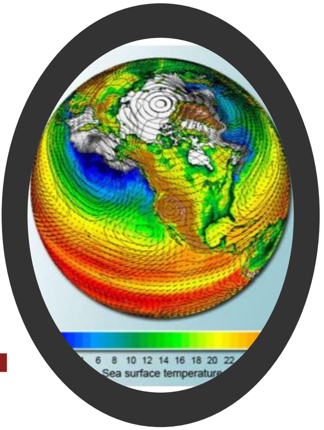
SCHOOL OF COMPUTING
UNIVERSITY OF UTAH

SOFTWARE ANALYSIS RESEARCH LABORATORY

ANALYSIS AND SYNTHESIS OF FLOATING-POINT ROUTINES

Zvonimir Rakamarić

FLOATING-POINT COMPUTATIONS ARE UBIQUITOUS



CHALLENGES

- FP is “weird”
  - Does not faithfully match math (finite precision)
  - Non-associative
  - Heterogeneous hardware support
- FP code is hard to get right
  - Lack of good understanding
  - Lack of good and extensive tool support
- FP software is large and complex
  - High-performance computing (HPC) simulations
  - Machine learning

FP IS WEIRD

- Finite precision and rounding
  - x + y in reals ≠ x + y in floating-point
- Non-associative
  - (x + y) + z ≠ x + (y + z)
  - Creates issues with
    - Compiler optimizations (e.g., vectorization)
    - Concurrency (e.g., reductions)
- Standard completely specifies only +, -, *, /, comparison, remainder, and square root
  - Only recommendation for some functions (trigonometry)

FP IS WEIRD cont.

- Heterogeneous hardware support
  - x + y*z on Xeon ≠ x + y*z on Xeon Phi
    - Fused multiply-add
  - Intel's online article "Differences in Floating-Point Arithmetic Between Intel Xeon Processors and the Intel Xeon Phi Coprocessor"
- Common sense does not (always) work
  - x "is better than" log(e^x)
  - (e^x-1)/x "can be worse than" (e^x-1)/log(e^x)
    - Error cancellation

FLOATING-POINT NUMBERS

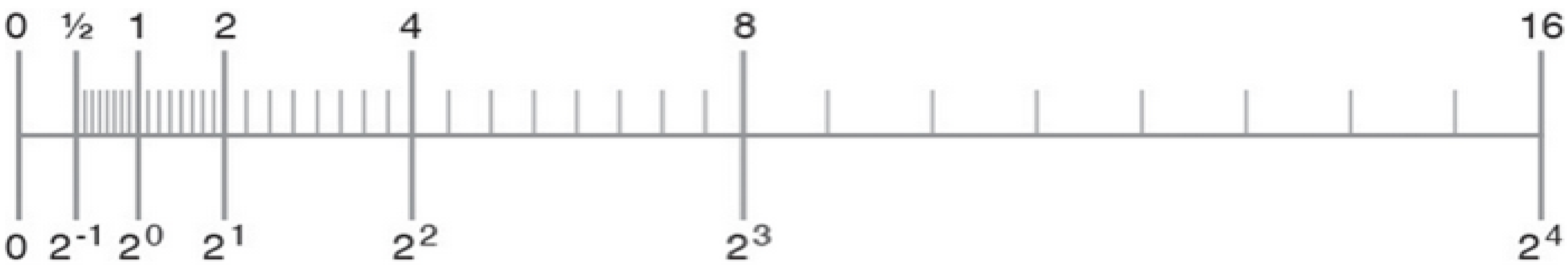
- IEEE 754 standard
- Sign (s), mantissa (m), exponent (exp):

$$(-1)^{s} * 1.m * 2^{exp}$$

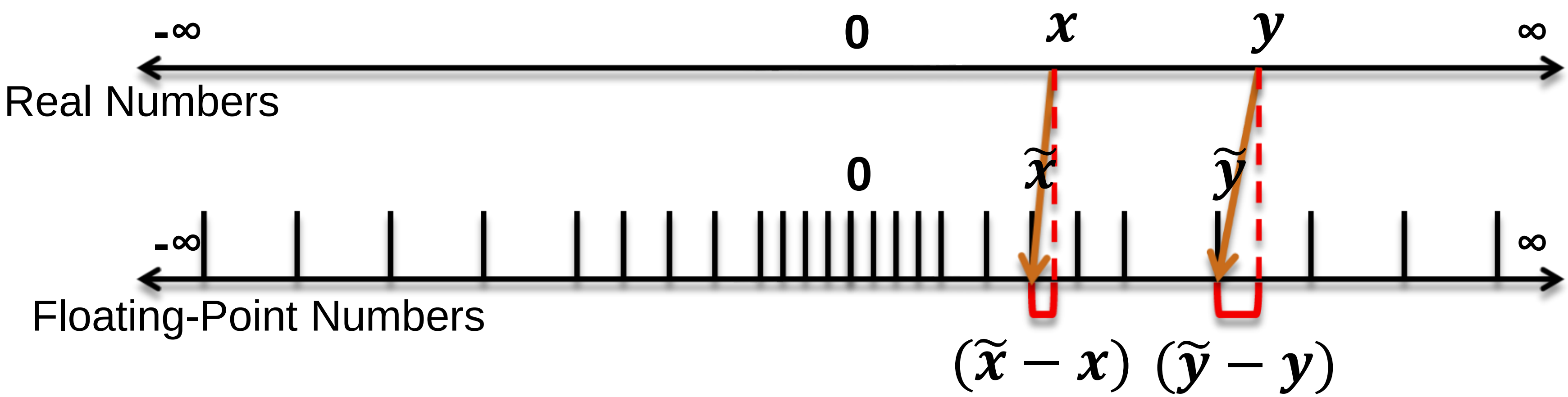
- Single precision: 1, 23, 8 bits
- Double precision: 1, 52, 11 bits

FLOATING-POINT NUMBER LINE

- 3 bits for precision
- Between any two powers of 2, there are $2^3 = 8$ representable numbers



ROUNDING IS SOURCE OF ERRORS



FLOATING-POINT OPERATIONS

- First normalize to the same exponent
  - Smaller exponent -> shift mantissa right
- Then perform the operation
- Losing bits when exponents are not the same!

UTAH FLOATING-POINT TEAM

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2. Zvonimir Rakamarić (prof)
3. Ian Briggs (staff programmer)
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Alumni: Alexey Solovyev (postdoc), Wei-Fan Chiang (PhD), Dietrich Geisler (undergrad), Liam Machado (undergrad)

RESEARCH THRUSTS

Analysis

- Verification of floating-point programs
- Estimation of floating-point errors
  1. Dynamic
     - Best effort, produces lower bound (under-approximation)
  2. Static
     - Rigorous, produces upper bound (over-approximation)

Synthesis

- Rigorous mixed-precision tuning

Constraint Solving

- Search-based solving of floating-point constraints
- Solving mixed real and floating-point constraints

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  2. **Static**
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Synthesis

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Constraint Solving

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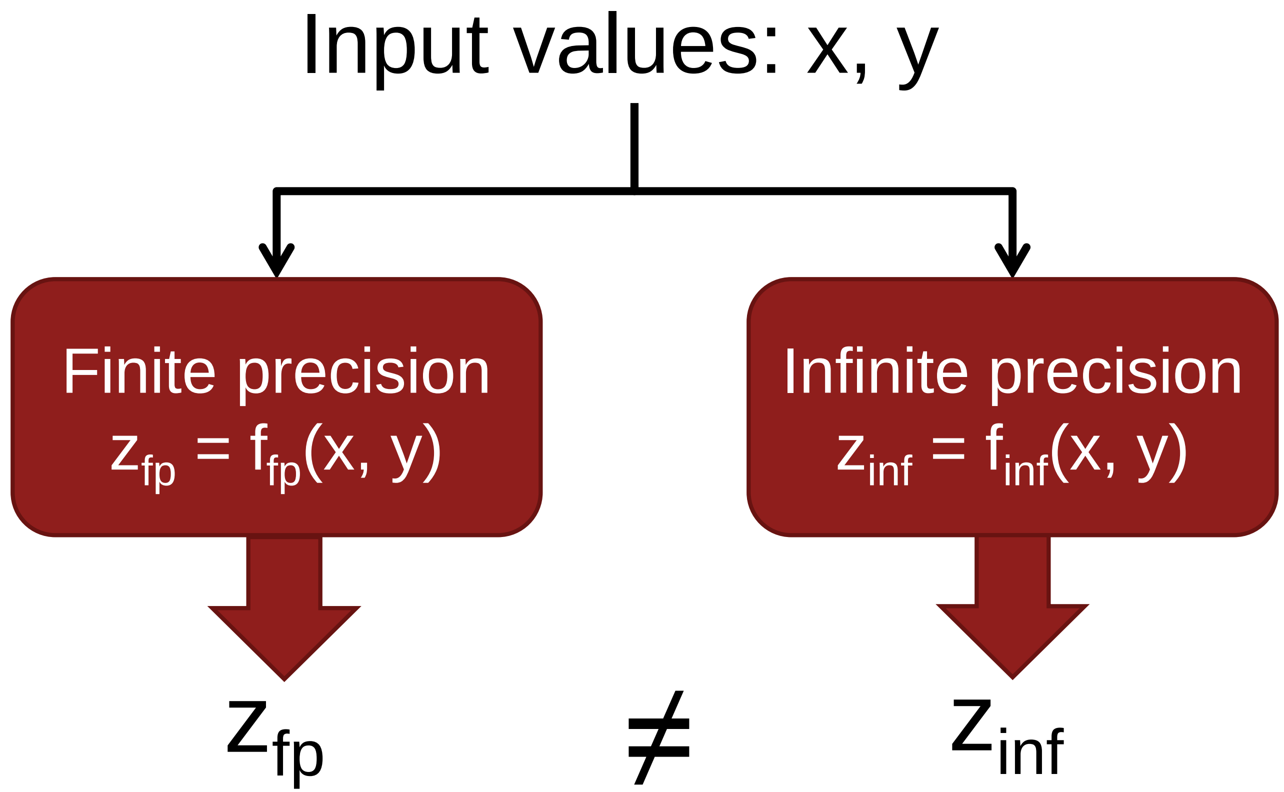
ERROR ANALYSIS

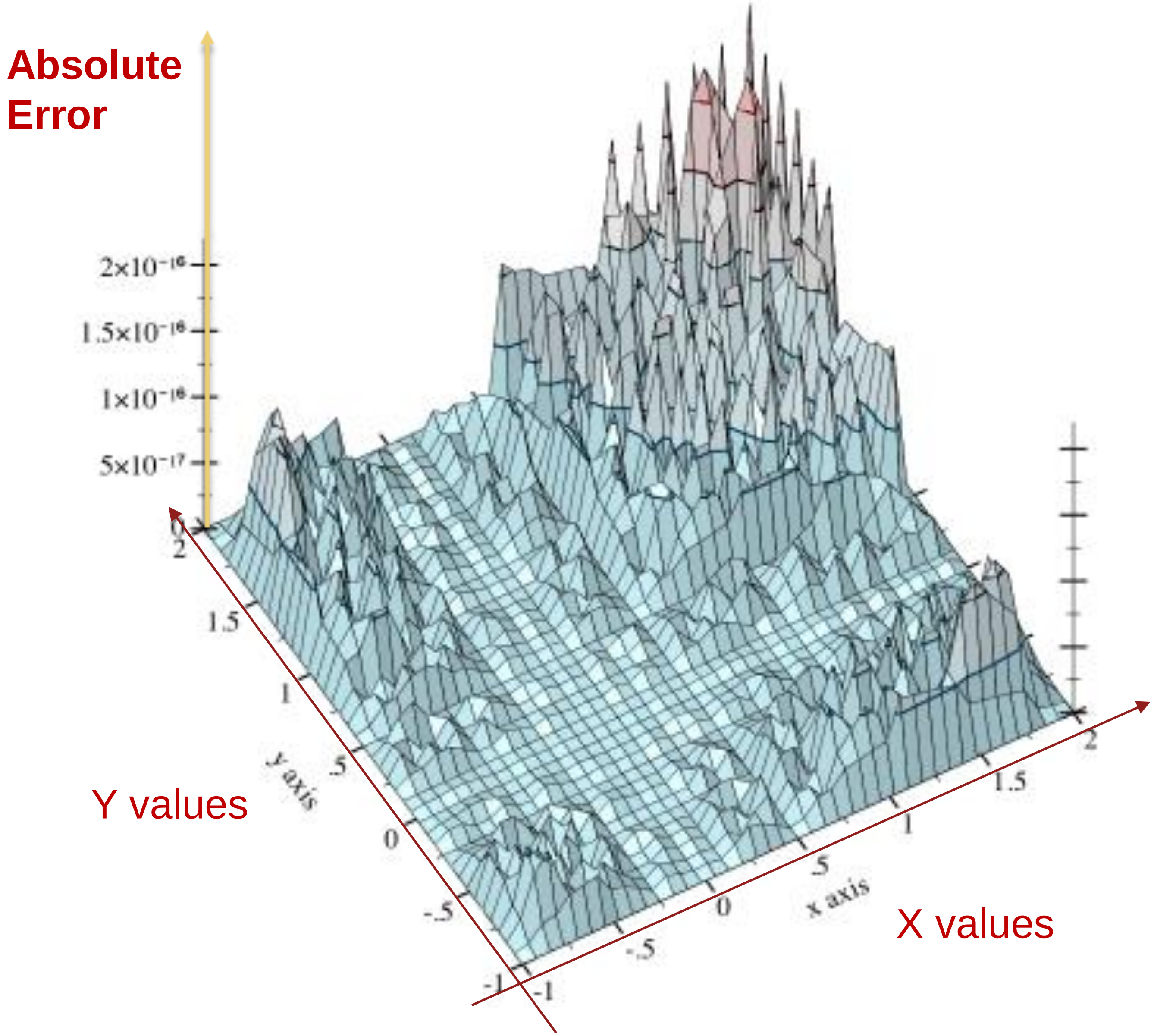
FLOATING-POINT ERROR



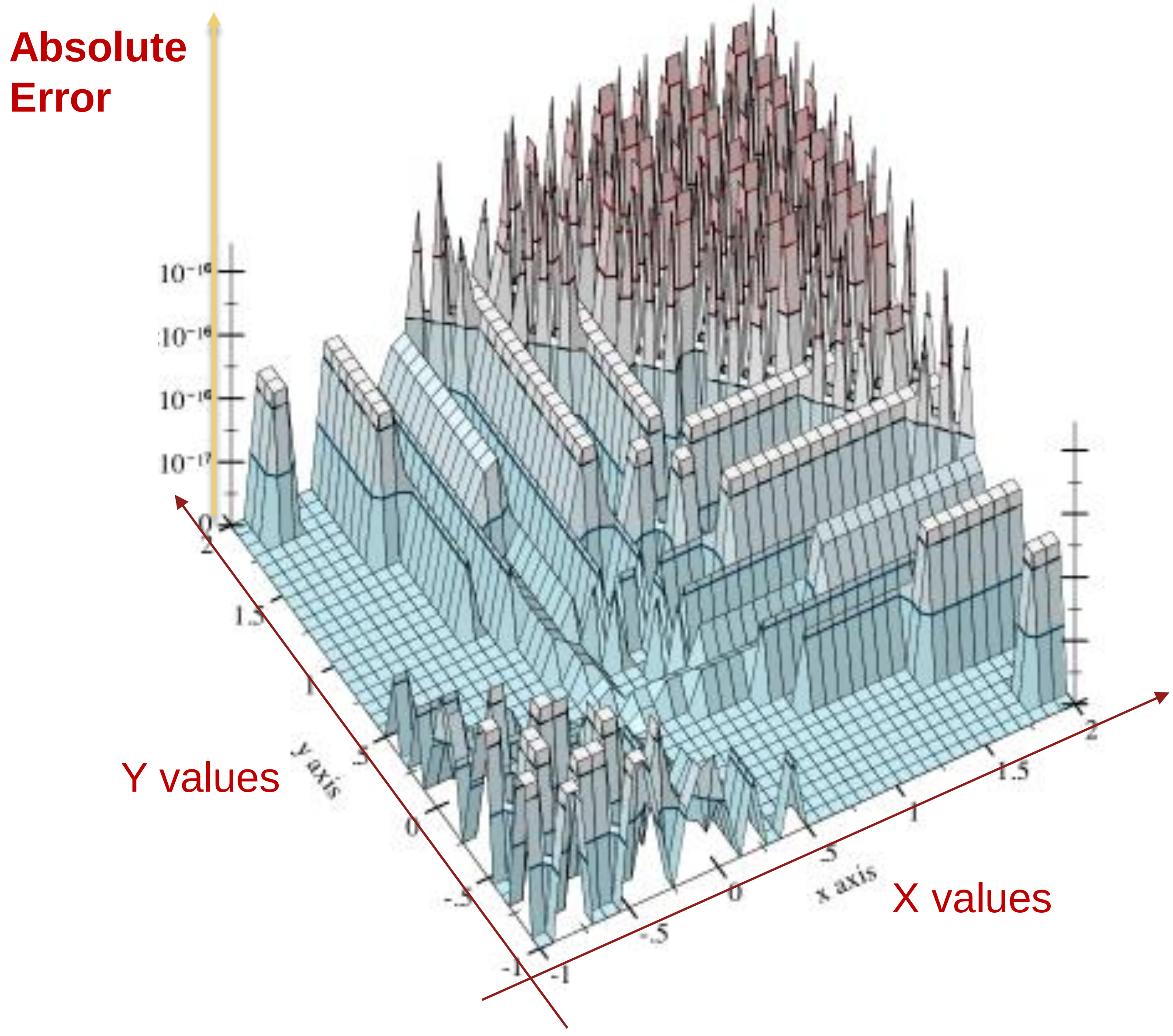
Absolute error: $| z_{fp} - z_{inf} |$

Relative error: $| (z_{fp} - z_{inf}) / z_{inf} |$

ERROR PLOT FOR MULTIPLICATION



ERROR PLOT FOR ADDITION



USAGE SCENARIOS

- Reason about floating-point computations
- Precisely characterize floating-point behavior of libraries
- Support performance-precision tuning and synthesis
- Help decide where error-compensation is needed
- “Equivalence” checking

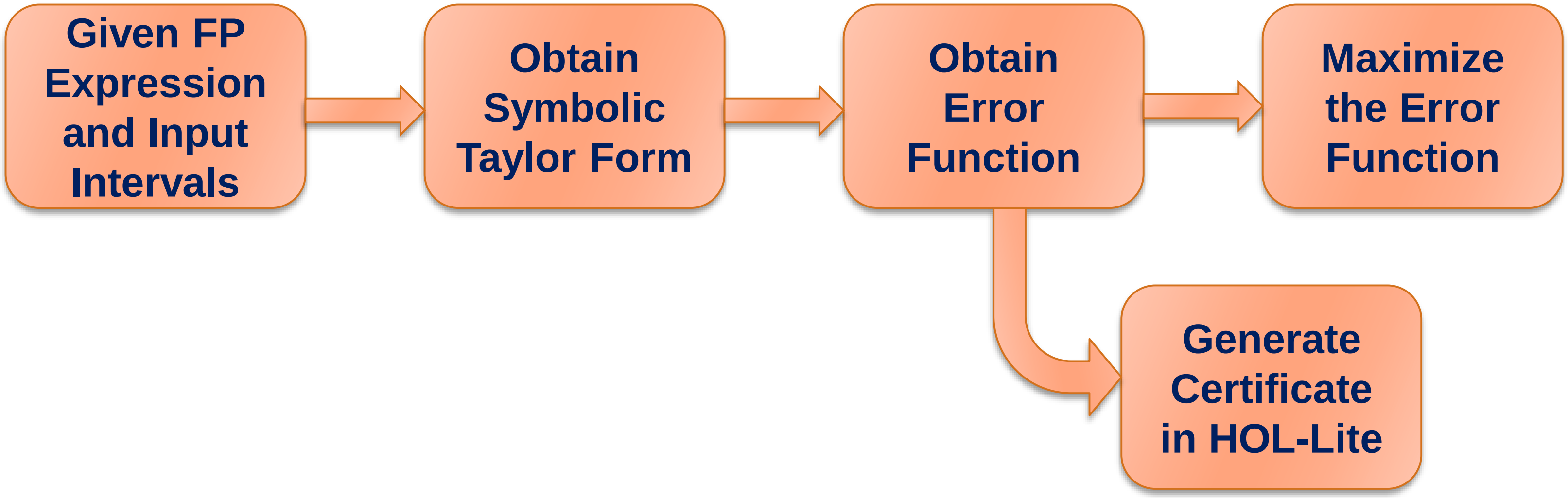
STATIC ANALYSIS

http://github.com/soarlab/FPTaylor

CONTRIBUTIONS

- Handles non-linear and transcendental functions
- Tight error upper bounds
  - Better than previous work
- Rigorous
  - Over-approximation
  - Based on our own rigorous global optimizer
  - Emits a HOL-Lite proof certificate
    - Verification of the certificate guarantees estimate
- Tool called FPTaylor publicly available

FPTaylor TOOLFLOW



IEEE ROUNDING MODEL

Consider $op(x, y)$ where x and y are floating-point values, and op is a function from floats to reals

IEEE round-off errors are specified as

$$op(x, y) \cdot \left(1 + e_{op}\right) + d_{op}$$

For normal values For subnormal values

Only one of e_{op} or d_{op} is non-zero:

$\left|e_{op}\right| \leq 2^{-24}, \left|d_{op}\right| \leq 2^{-150}$ (single precision)

$\left|e_{op}\right| \leq 2^{-53}, \left|d_{op}\right| \leq 2^{-1075}$ (double precision)

ERROR ESTIMATION EXAMPLE

- Model floating-point computation of $E = x/(x+y)$ using reals as

$$\tilde{E} = \frac{x}{(x+y)\cdot(1+e_1)}\cdot(1+e_2)$$

$$|e_1| \leq \epsilon_1, |e_2| \leq \epsilon_2$$

- Absolute rounding error is then $\left|\tilde{E} - E\right|$
- We have to find the max of this function over
  - Input variables x, y
    - Exponential in the number of inputs
  - Additional variables e_1, e_2 for operators
    - Exponential in floating-point routine size!

SYMBOLIC TAYLOR EXPANSION

- Reduces dimensionality of the optimization problem
- Basic idea
  - Treat each e as “noise” (error) variables
  - Now expand based on Taylor’s theorem
    - Coefficients are symbolic
    - Coefficients weigh the “noise” correctly and are correlated
- Apply global optimization on reduced problem
  - Our own parallel rigorous global optimizer called Gelpia
  - Non-linear reals, transcendental functions

ERROR ESTIMATION EXAMPLE

$$\tilde{E} = \frac{x}{(x+y)\cdot(1+e_1)}\cdot(1+e_2)$$

expands into

$$\tilde{E} = E + \frac{\partial \tilde{E}}{\partial e_1}(0) \times e_1 + \frac{\partial \tilde{E}}{\partial e_2}(0) \times e_2 + \mathrm{M}_2$$

where M_2 summarizes the second and higher order error terms and $|e_0| \leq \epsilon_0, |e_1| \leq \epsilon_1$

Floating-point error is then bounded by

$$\left|\tilde{E} - E\right| \leq \left|\frac{\partial \tilde{E}}{\partial e_1}(0)\right| \times \epsilon_1 + \left|\frac{\partial \tilde{E}}{\partial e_2}(0)\right| \times \epsilon_2 + \mathrm{M}_2$$

ERROR ESTIMATION EXAMPLE

- Using global optimization find constant bounds
- M_2 can be easily over-approximated
- Greatly reduced problem dimensionality
  - Search only over inputs x, y using our Gelpia optimizer

$$\forall x, y. \left|\frac{\partial \tilde{E}}{\partial e_1}(0)\right| = \left|\frac{x}{x+y}\right| \leq U_1$$

$$\left|\tilde{E} - E\right| \leq \boxed{\left|\frac{\partial \tilde{E}}{\partial e_1}(0)\right|} \times \epsilon_1 + \left|\frac{\partial \tilde{E}}{\partial e_2}(0)\right| \times \epsilon_2 + M_2$$

ERROR ESTIMATION EXAMPLE

- Operations are single-precision (32 bits)

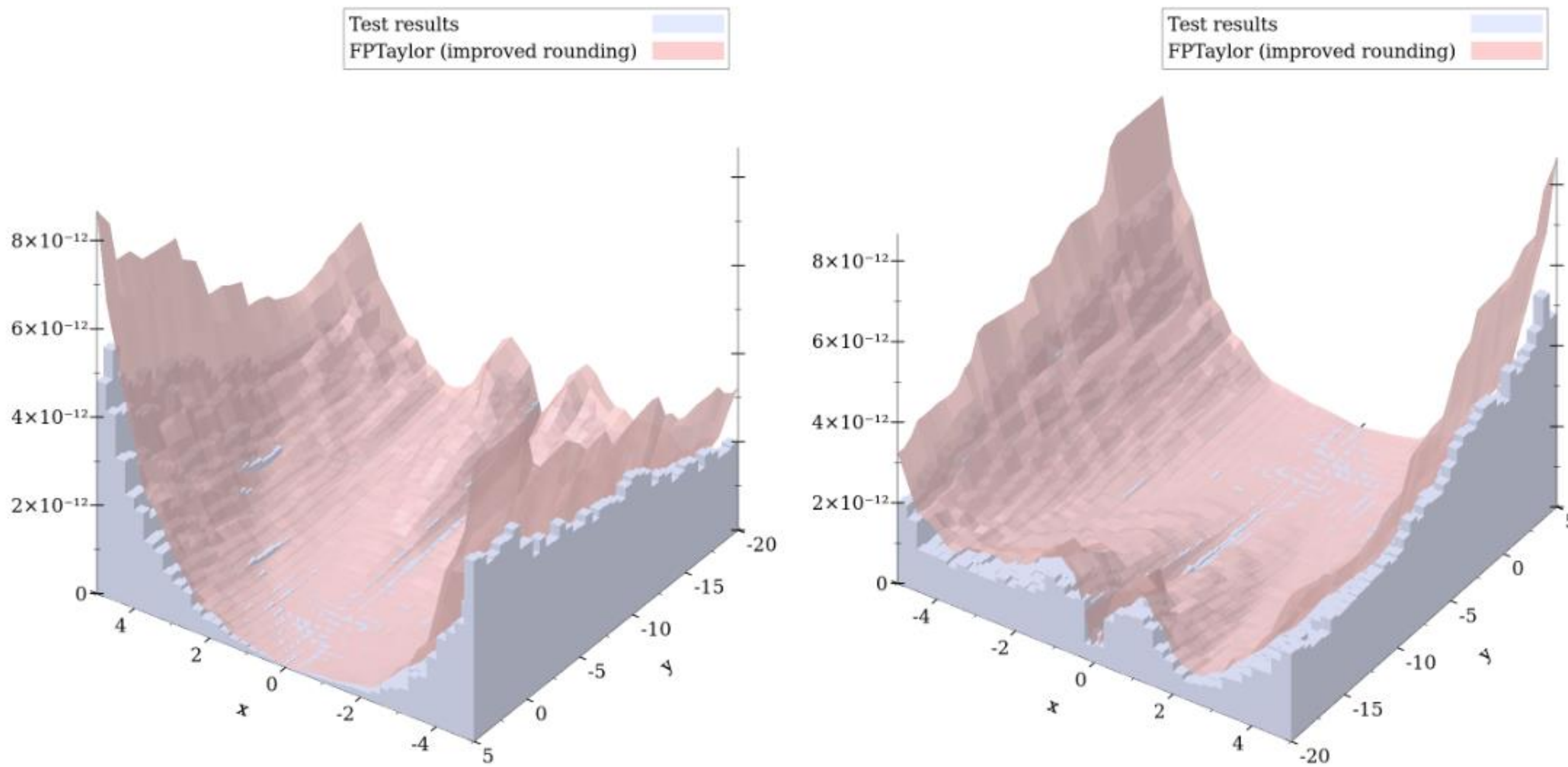
$$\left|\tilde{E} - E\right| \leq U_1 \times \epsilon_{32-bit} + U_2 \times \epsilon_{32-bit}$$

- Operations are double-precision (64 bits)

$$\left|\tilde{E} - E\right| \leq U_1 \times \epsilon_{64-bit} + U_2 \times \epsilon_{64-bit}$$

RESULTS FOR JETENGINE

jetEngine, $x_1 \in [-5, 5], x_2 \in [-20, 5]$, Double Precision



SUMMARY

- New method for rigorous floating-point round-off error estimation
- Our method is embodied in new tool FPTaylor
- FPTaylor performs well and returns tighter bounds than previous approaches

SYNTHESIS

http://github.com/soarlab/FPTuner

MIXED-PRECISION TUNING

Goal:
Given a real-valued expression and output error bound, automatically synthesize precision allocation for operations and variables

APPROACH

- Replace machine epsilons with symbolic variables

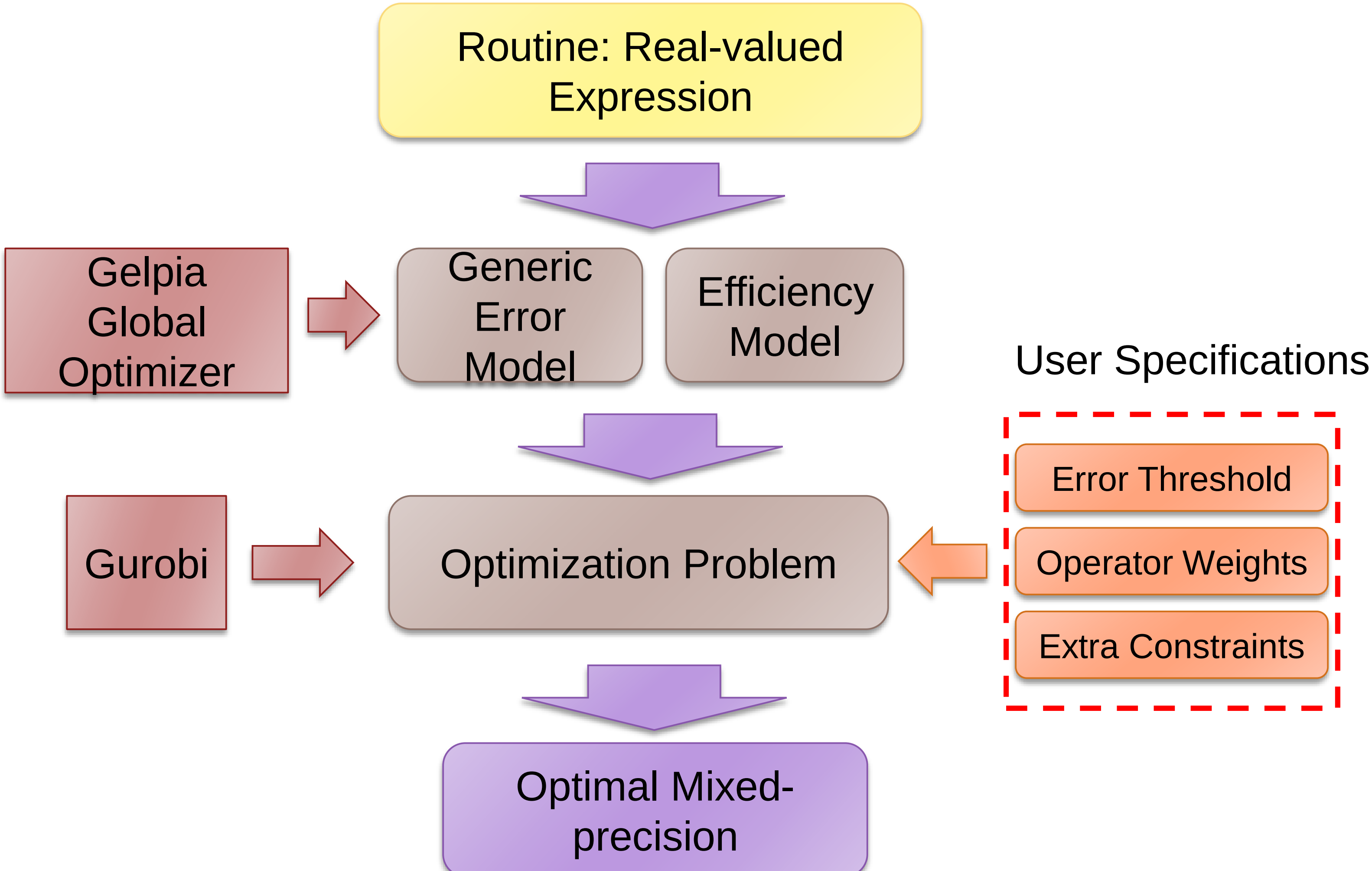
$$s_0, s_1 \in \{\epsilon_{32-bit}, \epsilon_{64-bit}\}$$

$$\left|\tilde{E} - E\right| \leq U_1 \times s_1 + U_2 \times s_2$$

- Compute precision allocation that satisfies given error bound
  - Take care of type casts

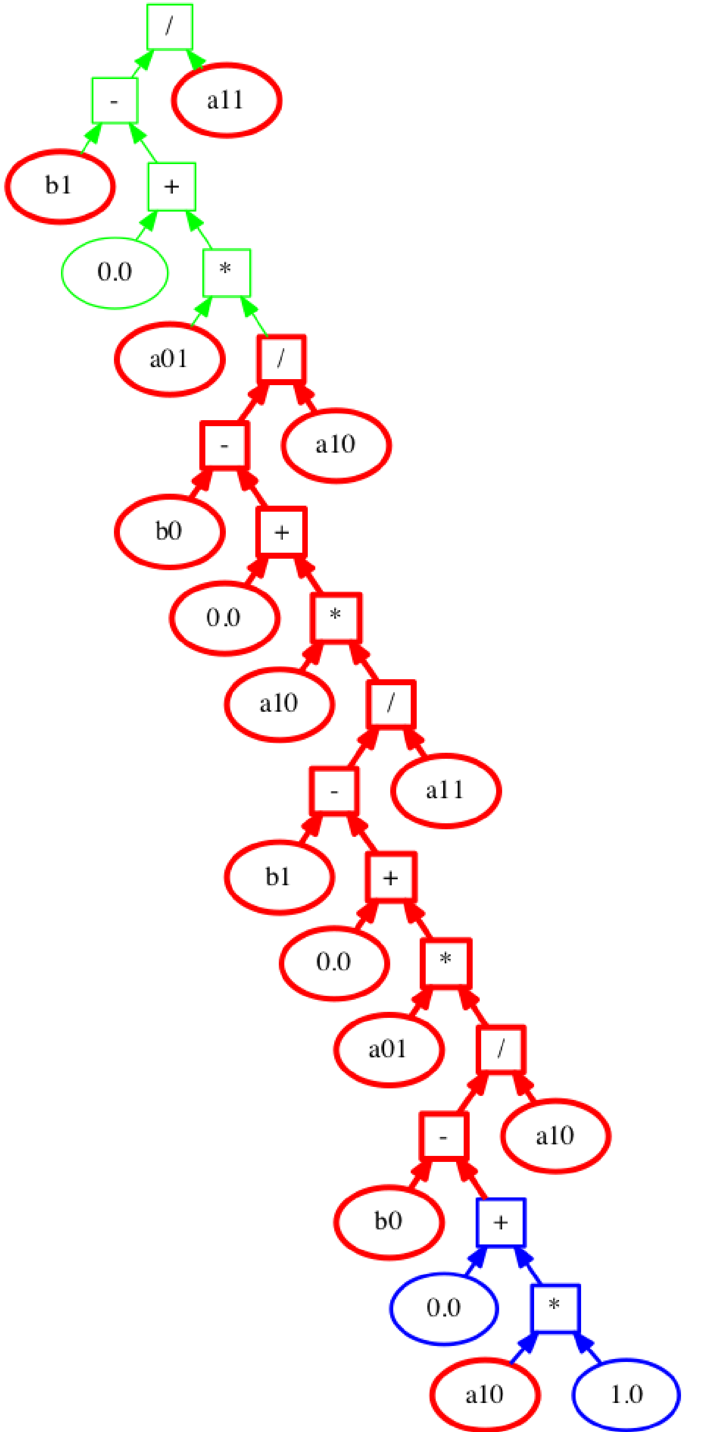
- Implemented in FPTuner tool

FPTuner TOOLFLOW



EXAMPLE: JACOBI METHOD

- Inputs:
  - 2x2 matrix
  - Vector of size 2
- Error bound: 1e-14
- Available precisions: single, double, quad
- FPTuner automatically allocates precisions for all variables and operations



SUMMARY

- Support mixed-precision allocation
- Based on rigorous formal reasoning
- Encoded as an optimization problem
- Extensive empirical evaluation
  - Includes real-world energy measurements showing benefits of precision tuning

SOLVING

http://github.com/soarlab/OL1V3R

MOTIVATION

- Poor scalability of floating-point solvers
  - Bit-blasting: formula → circuit

- Others showed that search-based solving can be effective for various SMT theories
  - Perform the search directly on theory level

- Can we achieve similar efficiency using stochastic local search on floating-points?
  - Inspired by Z3's qfbv-sls tactic for bit-vectors

STOCHASTIC LOCAL SEARCH

- Basic setting: local search + random choices

- Key ingredients
  - Score function
  - Neighborhood relation
  - Heuristics

SCORE FUNCTION

- score(expr, assignment) → rational

- Intuition: the ``degree'' of satisfiability
  - 1 = satisfiable
  - Example: s(x>2, x←1.99) > s(x>2, x←0)

- Key idea: measure a distance between *signed ordinal indices* of two floats
  - Total order on floats
  - Neighboring floats have a distance of 1

NEIGHBORHOOD RELATION

- Define neighbors of an assignment in a search step

- Several allowed mutations
  - Bit-flipping
  - ±ulp
  - (*2), (/2) – changing exponent

HEURISTICS

- Remove equality constraints when possible
  - (assert (and (= x (+ y z)) (> x 2.0)))

    → (assert (> (+ y z) 2.0))
- Use models derived from real arithmetic as initial assignments
  - (assert (> (+ y z) 2.0)) → y = 1, z = 3/2
- Variable neighborhood search
  - Refine the neighborhood relation into 3 subgroups and switch them on the fly

EVALUATION

- Compare OL1V3R with 5 state-of-the-art floating-point solvers

Tool	Version	Technique
MathSAT	5.5.4	Hybrid
CVC4	1.7	Bit-blasting
Z3	4.8.4	Bit-blasting
JFS	commit 2322167	Coverage-guided fuzzing
COLIBRI	revision 2176	Constraint propagation

RESULTS

Tool	Sat	Unsat	Unknown	Timeout	$Diff^B$	$Diff^H$
$OL1V3R^B$	115	0	2	80	-	0/16
$OL1V3R^H$	131	0	2	64	16/0	-
MathSAT	125	1	7	64	13/5	2/9
CVC4	117	1	10	69	10/9	2/15
Z3	88	0	10	99	3/32	0/43
JFS	113	0	0	84	4/8	0/20
COLIBRI	118	32	4	43	14/13	3/18

SUMMARY

- Implemented a prototype for solving floating-point constraints using SLS
  - Define key ingredients (score function, neighbors)
  - Devise custom heuristics

- Compared our tool to state-of-the-art solvers and confirmed its effectiveness