Exercise 03 - Solution

Box, MILP and DeepPoly Certification

Reliable and Trustworthy Artificial Intelligence
ETH Zurich

Solution 1.

(a) It is $[0,1] -^\sharp [0,1] = [0,1] +^\sharp [-1,0] = [-1,1]$. However, the only possible result of $x - x$ for $x \in [0,1]$ is 0, which could be represented exactly using the more precise interval $[0,0]$.

(b) This is true. Let $x \in [a,b]$ and $y \in [c,d]$. Hence, $a \leq x \leq b$ and $c \leq y \leq d$. Now consider the sum $z := x + y$. It is $a + c \leq z$ and $z \leq b + d \leq b + |d|$, because $d \leq |d|$. Therefore, $z \in [a + c,\ b + |d|]$.

(c) This is not true. Consider $[-1,-1] +'' [0,0] = [-\infty, -2]$, which does not include value -1 produced by $-1 + 0$.

(d) Recall that $a \leq b$. We have to distinguish the three general positions of the interval w.r.t. 0: below ($b \leq 0$), above ($a \geq 0$), or including 0 (otherwise).

$$[g,h] = f^\sharp([a,b]) = \begin{cases} [a^2, b^2] & \text{if } a \geq 0, \\ [b^2, a^2] & \text{else if } b \leq 0, \\ [0, \max(a^2, b^2)] & \text{otherwise.} \end{cases}$$

(e) The most precise sound transformer is:

$$[a,b] \cdot^\sharp [c,d] \ = \ [\min(ac, ad, bc, bd),\ \max(ac, ad, bc, bd)]$$

Note: To see this, perform a case distinction on the signs of a, b, c, and d. The naive solution attempt $[ab, cd]$ is unsound (why?).

(f) The most precise sound transformer is:

$$[y_1, y_2] = \max{}^\sharp([a_1, b_1], [a_2, b_2]) = [\max(a_1, a_2),\ \max(b_1, b_2)].$$

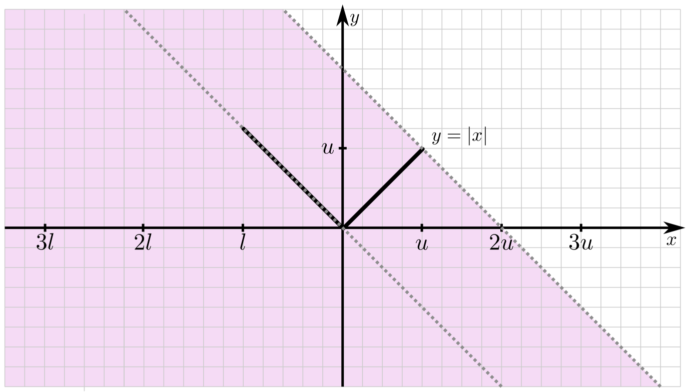
Solution 2. The intervals for the different neurons in the network are:

$$\begin{aligned} x_1 &\in [0,1] & x_6 &\in [0,1] \\ x_2 &\in [0,1] & x_7 &\in [0,2] \\ x_3 &\in [0,2] & x_8 &\in [0,1] \\ x_4 &\in [-2,-1] & x_9 &\in [0,2] \\ x_5 &\in [-1,1] & x_{10} &\in [-2.5, 0.5] \end{aligned}$$

From this, we cannot conclude that $x_9 > x_{10}$. In particular, the lower bound for $x_9 - x_{10}$ is -0.5, which is not sufficient to prove the property.

Solution 3.

(a) See the following figure.



(b) We can use an analogous construction as in the first sub-question to create a line which (i) for $a = 1$ coincides with the line segment at $x \geq 0$, and (ii) for $a = 0$ matches the lower end of the line segment at $x = l$. To this end, we construct the following inequality constraints, which bound the values of y from above:

$$\frac{y}{2} \leq \frac{x}{2} - l \cdot (1 - a), \qquad a \in \{0, 1\}$$

As a last step, we also need to bound y from below according to the "v-shape" of the absolute function. This can easily be achieved using the constraints

$$y \geq x \quad \text{and} \quad y \geq -x.$$

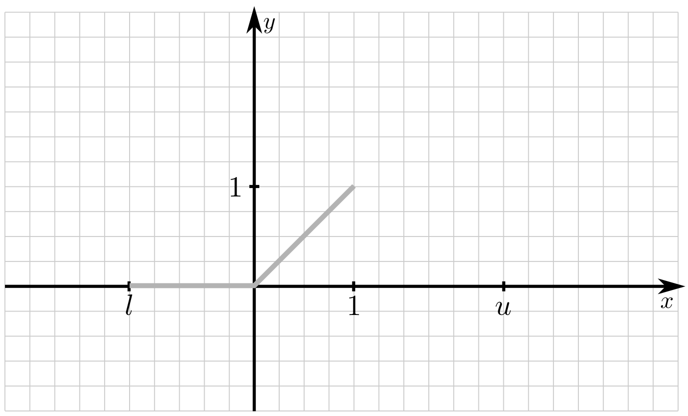
Combining these items leads to the following set of MILP constraints, which exactly represents the bold line segments in the figure.

$$\begin{aligned}
&\frac{y}{2} \leq \frac{x}{2} - l \cdot (1-a),\\
&y \geq x,\\
&\frac{y}{2} \leq -\frac{x}{2} + u \cdot a,\\
&y \geq -x,\\
&a \in \{0, 1\}.
\end{aligned}$$

Note: It may be tempting to use products $a \cdot x$ between a and x in order to obtain an arguably simpler constraint set. However, such solutions are invalid as they are not linear in the variables.

Solution 4.

(a) See the following figure.



(b) The missing inequalities are:

$$\begin{array}{ll}
\text{(i)} & y \geq b \\
\text{(ii)} & y \geq x + (1-u) \cdot b
\end{array}$$

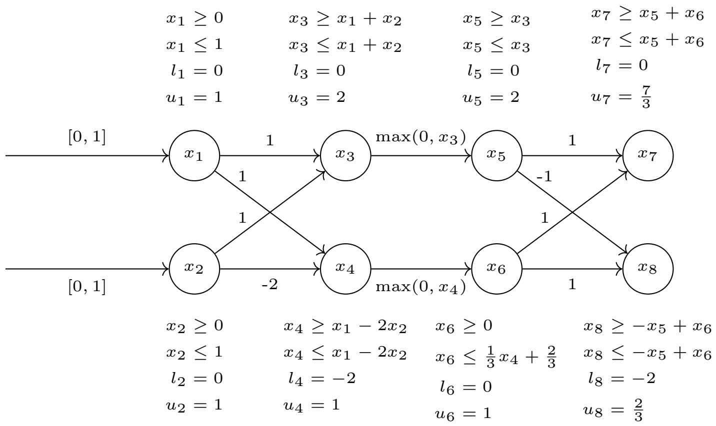
For $b = 0$, this reduces to the system in subtask (a). For $b = 1$, it must be $a = 1$ and the system reduces to $y = 1,\ x \in [1, u]$.

Note: It may be tempting to use products $a \cdot x$ between a (or b) and x in order to obtain an arguably simpler constraint set. However, such solutions are invalid as they are not linear in the variables.

Solution 5. (a) The area for Option 1 is $A_1 = u\frac{(u-l)}{2}$, while the area for Option 2 is $A_2 = -l\frac{(u-l)}{2}$. Hence, we should pick Option 1 if

$$\begin{aligned} A_1 &\le A_2 \\ \iff u\frac{(u-l)}{2} &\le -l\frac{(u-l)}{2} \\ \iff u &\le -l \end{aligned}$$

(b) The figure below shows the result of our analysis.



For the ReLUs, we used that x_3 is strictly positive and that x_4 satisfies $-l_4 \ge u_4$ (hence we used Option 1).

To compute the lower and upper bounds, we computed the following:

$$
\begin{aligned}
& x_1 \geq 0 =: l_1 \\
& x_1 \leq 1 =: u_1 \\
& x_2 \geq 0 =: l_2 \\
& x_2 \leq 1 =: u_1 \\
& x_3 \geq x_1 + x_2 \geq 0 + 0 = 0 =: l_3 \\
& x_3 \leq x_1 + x_2 \leq 1 + 1 = 2 =: u_3 \\
& x_4 \geq x_1 - 2x_2 \geq 0 - 2 \cdot 1 = -2 =: l_4 \\
& x_4 \leq x_1 - 2x_2 \leq 1 - 2 \cdot 0 = 1 =: u_4 \\
& x_5 \geq x_3 = x_1 + x_2 \geq 0 + 0 =: l_5 \\
& x_5 \leq x_3 = x_1 + x_2 \leq 1 + 1 =: u_5 \\
& x_6 \geq 0 =: l_6 \\
& x_6 \leq \tfrac{1}{3}x_4 + \tfrac{2}{3} \leq \ldots \text{(as above)} \leq \tfrac{1}{3} \cdot 1 + \tfrac{2}{3} = 1 =: u_6 \\
& x_7 \geq x_5 + x_6 \geq x_3 + 0 \geq \ldots \text{(as above)} \geq 0 := l_7 \\
& x_7 \leq x_5 + x_6 \leq x_3 + \tfrac{1}{3}x_4 + \tfrac{2}{3} \leq x_1 + x_2 + \tfrac{1}{3}(x_1 - 2x_2) + \tfrac{2}{3} = \tfrac{4}{3}x_1 + \tfrac{1}{3}x_2 + \tfrac{2}{3} \leq \tfrac{7}{3} := u_7 \\
& x_8 \geq -x_5 + x_6 \geq -x_3 + 0 \geq \ldots \text{(as above)} \geq -2 := l_8 \\
& x_8 \leq -x_5 + x_6 \leq -x_3 + \tfrac{1}{3}x_4 + \tfrac{2}{3} \leq -(x_1 + x_2) + \tfrac{1}{3}(x_1 - 2x_2) + \tfrac{2}{3} \\
& \quad = -\tfrac{2}{3}x_1 - \tfrac{5}{3}x_2 + \tfrac{2}{3} \leq \tfrac{2}{3} =: u_8
\end{aligned}
$$

Using the analysis result, we can show that

$$x_7 - x_8 \geq x_5 + x_6 - (-x_5 + x_6) = 2x_5 \geq \ldots \text{(as above)} \geq 0. \tag{1}$$

Note that we perform symbolic simplifications during back-substitution whenever possible. For example, in Eq. (1), we simplified $x_6 - x_6$ to 0. Without these critical simplifications, we get a worse lower bound:

$$
\begin{aligned}
x_7 - x_8 &\geq x_5 + x_6 - (-x_5 + x_6) \\
&= x_5 + x_6 + x_5 - x_6 \\
&\geq x_3 + 0 + x_3 - \left(\tfrac{1}{3}x_4 + \tfrac{2}{3}\right) \\
&= x_3 + x_3 - \tfrac{1}{3}x_4 - \tfrac{2}{3} \\
&\geq x_1 + x_2 + x_1 + x_2 - \tfrac{1}{3}(x_1 - 2x_2) - \tfrac{2}{3} \\
&\geq x_1 + x_2 + x_1 + x_2 - \tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 - \tfrac{2}{3} \\
&\geq 0 + 0 + 0 + 0 - \tfrac{1}{3} + 0 - \tfrac{2}{3} = -1
\end{aligned}
$$